THE

THREE FIRST SECTIONS

AND

PART OF THE SEVENTH SECTION

OF

NEWTON'S PRINCIPIA,

WITH A PREFACE

RECOMMENDING

A GEOMETRICAL COURSE OF MATHEMATICAL READING,

AND AN INTRODUCTION

ON THE ATOMIC CONSTITUTION OF MATTER, AND THE
LAWS OF MOTION.

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PREFACE.

THE recent alterations in the Examining Statute will give, we may hope, a powerful impulse to studies in Physical knowledge. Let us therefore consider what branches in that department seem best adapted to forward the great objects proposed in an Academic education. The study of any branch of Physical knowledge forms a healthful exercise of the mental powers, and supplies the mind with interesting information; and, stimulating to continued thought and observation, calls into action those purer intellectual energies, which, in proportion to their perfection and intensity, constitute the happiness of man, and elevate his taste and conceptions above the low range of sensual gratifications. But those habits of close attention, abstraction, and patient investigation, which an University education is expected to generate, are far more likely to be assisted and strengthened by studies of Physical Science, grounded on systems of strict reasoning and connected argument, than by showy attainments in the light superficial departments. Some acquaint-

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ance with the general principles of Chemistry,
with the all-pervading agencies and the general
laws established by it, appears a necessary first
step in Physical researches, somewhat in the man-
ner of the Grammar and Dictionary being required
in attaining a knowledge of Latin or Greek.
Superior encouragement will, I trust, be held
out to the four regular branches of Natural Phi-
losophy, from the superior value of the knowledge
they convey, and the strict reasoning and mathe-
matical demonstration by which that knowledge is
attained. I shall therefore venture an observation
on the line of argument, or reasoning, by which in
many cases we must be content to reach our con-
clusions. Though sincerely rejoicing that the
Book of Nature will no longer be a closed volume
to many of our Students, we must remember the
paramount importance of a correct acquaintance
with the Book of Grace, and of a disposition to
acquiesce in its revelations. We must, therefore,
not wish those studies in ancient Classical Lite-
rature to be in any way superseded, which have
hitherto constituted the main feature in an Oxford
education ; I mean, more particularly, an intimate
acquaintance with the History of Thucydides, and
the Ethical Works of Aristotle. In those Ethical
Works, an exhibition is laid open of the governing
principles, actuating motives, and prevailing habits
of man's moral constitution, more full, accurate,
and minute, than is to be found in any other work
of any age or nation. The Student, regularly and

closely catechized in the facts there detailed, in the observations and sentiments arising from such details, also in long elaborate inductions from the sayings and practice of the wise, and in the conclusions grounded on such inductions, seems admitted, day after day, to a dissection of man's moral constitution. The constituent parts, one after another, are laid open to his view. He soon realizes the general theory, that habit becomes nature, and that continued acts form the habit. He is taught that the perfection of man's character is not to be sought by smothering or eradicating the feelings and passions implanted in our nature, but by their being limited and controlled according to the dictates of right reason, and that their action, when thus habitually controlled, constitutes the series of moral virtues. Whereas their action, when falling short, or exceeding, the standard pointed out by reason, gives a double corresponding series of moral vices. Now, the origin of infidelity and scepticism in religion may be traced up in great measure to an undue estimate of the powers of this governing principle of reason, both as to its general capability of pointing out the things and objects most essential to our real happiness, and as to its real efficiency in controlling and influencing the lower principles of our nature. Against any such overweening estimate, our Student is guarded by the information that, in the great majority of cases, reason is unable to enforce the due subordination of the appetites and

passions to its dictates, so that the individual, against his better sense and will, is often hurried into blameable and mischievous actions and indulgences. And, moreover, that the continuance of this insubordination superinduces a sad cloudiness of view, a sad vitiation of perception in reason itself. This light within us becomes darkness, and our moral judgments and convictions become perverted and depraved. These views and principles he finds exemplified and realized in the faithful and elaborate details given by Thucydides of the wars, convulsions, and revolutions of the several Grecian Commonwealths during a period of twenty years, in which Commonwealths the freedom in the forms of their government, and the rivalry of contending factions, awakened and called into full action all the energies of a highly talented and civilized race, so that the productions of that period in Literature and the Fine Arts are models to all succeeding ages. But in his narrative of the public events, the sage historian frequently comments on the sad innate depravity of human nature, which, when the violence and triumph of faction, whether democratic or aristocratic, had overborne for a time all restraints of law, rejoiced in the opportunity of exhibiting its inborn ἐπιχαιρεκακία, by openly disclaiming all regard to any moral sense of equity, and by the reckless indulgence of cruelty and licentiousness. When heathen philosophy and heathen history supply these proofs of the necessity of a supernatural

agency to purify our nature, can we help saying to the Philosophers and to the Elders in Church or State, who are sceptical on this head, " Art thou a master in Israel, and knowest not these things ?" and the following words of our Lord, " If I have told you earthly things, and ye believe not, how shall ye believe, if I tell you of heavenly things ?" lead us to anticipate in such persons, a real, though perhaps a covert, disbelief in the higher mysteries of Christianity. But our Oxford student, enlightened by his course of study as to the real working of man's moral constitution, sees in it the ruins of an original far better fabric, where by some shock a disorganization or disturbance in the original harmony of the constituent parts has taken place. Consequently his mind, thoroughly initiated in this view, will on opening the Scriptures see no stumblingblock in the tidings of Adam's fall, and in the Gospel doctrines of Redemption, Atonement, and Sanctification. And the experience of subsequent years, grounded on observation of what is passing around him and on consciousness of what is passing within him, will expand and strengthen his conviction, that these Gospel doctrines set forth a dispensation, well worthy of Divine Love and Divine Wisdom: inasmuch as means of supernatural purification and strength are there disclosed for remedying our innate corruption, invigorating the infirm energies of our reason, and carrying on its view to those most awful sanctions of eternity, which alone have power to enlist our instincts of

self-love in the cause of piety and virtue. Our
severe, well-selected course of ancient Classical
lore, prepossessing the mind with a correct humi-
liating sense of man's moral constitution, does thus
dispose us to a willing thankful acquiescence in
the truths and mysteries of Revelation : and our
humbling conviction, how open our reason is to
prejudices, conceit, and perversion, will incline us
to investigate and adhere to that sense and in-
terpretation of Scripture, which has a stamp of
Apostolic authority, from its accordance with the
Creeds and teaching of the Primitive Christian
Church. Much time and attention must be absorbed
in these valuable and indispensable Classical studies :
and comparatively but few Students will possess
energy and talent sufficient to combine high
attainments in these studies, with such a progress
in the higher Mathematical Analytics, as to be
competent to carry on researches in Physical know-
ledge by aid of the Differential Calculus. We must
therefore be content in the great majority of cases
to prosecute these researches by aid of geometrical
reasonings, and accordingly in such cases to limit
our pure Mathematics to the four first, and sixth
books of Euclid. Algebra to the end of Quadratic
Equations, the general principles of Trigonometry,
the general properties of the Conic Sections : and
to crown these very moderate attainments with a
knowledge of the three first Sections of Newton's
Principia. A moderate weekly portion of time and
attention would suffice for mastering this limited

range of Mathematical knowledge, and would not be any unfair encroachment on the hours and energies required for Classical studies. Moreover, the quiet view, thus leisurely carried on in these less abstruse Geometrical departments, would be attended throughout its range with full satisfactory conviction. The mind would clearly see how each successive step was built on a preceding step, and would be trained to such a habit of closely-connected reasoning, as might in many cases be an antidote to the unhealthy influence produced by the practice of hastily gulping down portions of knowledge, which the mind does not properly digest, and accordingly no benefit, but rather a nausea, is produced thereby. In the cases I am contemplating, if the pure Mathematics were to comprehend the Differential Calculus as the instrument for subsequent Physical enquiries, the students from deficient opportunity would frequently not attain any clear satisfactory conceptions of the Analytical principles and processes, and would be discouraged by the apparent air of mystery hanging over them, from the further pursuit of Mathematical or Physical studies. Looking round on the phenomena of nature, they might feel a strong desire for some well-grounded acquaintance with these interesting departments, Astronomy, Optics, Hydrostatics ; and we may suppose them uttering the wish, (Georgics, ii. 475.)

> Me verò primum dulces ante omnia Musæ,
> Accipiant, cælique vias et sidera monstrent:
> Defectus solis varios, lunæque labores.

Let therefore no such obstacle meet them on the threshold, as may deter them from further prosecution of studies, which would now form a healthful interesting exercise of their mental powers, and would generate a taste for such researches and observations, as would furnish a very pleasing and improving occupation for the leisure hours of after life; hours too sure, without such taste, to be given up to trifling amusements.

If the Fluxional analysis be made the only road for prosecuting Physical researches, how many would add,

> Sin, has ne possim naturæ accedere partes,
> Frigidus obstiterit circum præcordia sanguis;
> Flumina amem silvasque inglorius.

Look to the advantages derived from the study of Natural Philosophy, that it "improves and elevates the mind, by unfolding to it the magnificence, the order, and the beauty manifested in the construction of the material world: and that it offers the most striking proofs of the beneficence, the wisdom, and the power of the Creator." Look again to the limited opportunities of our Students for such study. The inference follows, that the Mathematical basis on which such study must stand to make it sound and satisfactory, need only contain such principles and portions of Mathematical knowledge, as are necessary to constitute a safe and complete basis.

INTRODUCTION.

MATTER presents itself to our view under three modifications, apparently very different as to internal constitution. We might at first suppose, that this seemingly great difference of constitution in bodies solid, liquid, and aeriform, indicated some essential difference in their elementary component particles. But turning our attention to the familiar processes going on over our kitchen fire, we there see a solid body, under the agency of heat, melted, i. e. passing into a liquid state, and we see the liquid body soon boiling, and emitting a sensible steam or vapour, i. e. passing from a liquid state into vapour, into an imperfect aeriform state. Thus we see that the action of heat easily causes the transition of a body from a solid to a liquid, and from a liquid to an aeriform state. Also the converse is familiar to our observation. By some cooling process, i. e. by the abstraction of heat, we see steam brought instantaneously back to a liquid state; and by freezing, i. e. by the continued abstraction of heat, we see water become ice, we see the liquid brought into the solid state.

Hence we infer, that no essential difference in their elementary component particles is necessarily indicated between bodies aeriform, liquid, and solid: this difference of internal structure only implies a greater or less infusion of heat. But according to this view, ought not a body in passing from solid to liquid, and from liquid to the aeriform state, in each transition to affect us with a more powerful sensation of heat? whereas we find ice giving 32° by Fahrenheit's scale, and that we may have water in the fluid state giving also 32°: moreover that boiling water gives 212°, and steam indicates by the thermometer no higher temperature. This brings an important and mysterious fact to light, that when additional heat is so infused into a body as to cause this characteristic change of structure, its action seems absorbed in producing this effect, so that the body gives out no external indication of having become more hot. Hence the heat producing this effect is called latent, inasmuch as it exists in the body in a latent state, as far as our own sense of feeling is concerned. The wonderfully minute particles, into which we can by processes of art divide matter, and again the still more extraordinary minuteness of this division, which we see carried on in the operations of nature, gave rise to the dispute, whether matter was or was not infinitely divisible. A general acquiescence seems now to prevail in the Atomic Theory, in the opinion that in thus subdividing any portion of matter into smaller and

smaller parts, you would ultimately come to par-
ticles admitting of no further subdivision, and
hence called atoms. But we must take up a notion
so far correct of their minuteness, as not to suppose
the dust or powder, into which we can reduce many
substances, to be the original elementary particles
of such substances, but that each grain of such
dust or powder is still divisible into portions
smaller and smaller, so as no longer to be recog-
nised by the eye without the aid of the microscope,
and that these particles, before they reach the limit
of subdivision, or atomic state, defy the power even
of the microscope to give them a dimension per-
ceptible to our vision.   Salt, sugar, and other sub-
stances may be so perfectly melted in water, as in
no degree to cloud the transparency of the water :
what is this but the division of the salt or sugar by
the solution into particles so extremely minute, as
to be invisible.   Our taste recognises their exist-
ence and diffusion throughout the water, but our
eye cannot perceive them.   Again, a small portion
of musk will powerfully scent the whole of a large
apartment.   This shews the diffusion of small par-
ticles of the musk through the whole area of the
apartment, because we perceive the scent by such
particles striking on our olfactory nerve.   But
what eye, what microscope can recognise these
floating particles ?   The animalcula, which the
microscope discloses flitting about in a tumbler of
water, the organization also of such animalcula, so
far ascertained, that we assign to them bones,

sinews, veins, blood, must impress us still more powerfully with the marvellous minuteness of atoms, or the ultimate particles of matter—a minuteness of which our present mental powers cannot form an adequate conception. The bodies around us are made up of these atoms, aggregated together in masses of various magnitudes and forms. In these masses the atoms are held toge- ther by an influence called attraction; which word implies, that atoms, whether separate or already joined into masses, tend towards all other atoms or masses, and with force in a certain proportion to their proximity. There are different kinds and modifications of this influence. We will first direct our attention to what is commonly called the Attraction of Cohesion, the attraction by which the particles of the same body cohere, or are bound together. There are numberless and wide differ- ences in the tenacity with which the particles of different bodies, and even of the same body under different circumstances, are cohering. This differ- ence of tenacity is principally owing to the different degrees of latent heat in the body. Heat infused into a body, separates the component particles, causing them to recede from one another, intro- ducing a new principle of action, called Repulsion. In the solid body the particles are adhering firmly together, and are not capable, while thus cohering, of being moved freely among one another, though they may by adequate force be violently rent or torn from one another. Let us now suppose fresh

and fresh additions of heat infused into the body. The body softens, i. e. the proximity of the atoms lessens, and therefore they attract one another, or cohere with less force. Suppose the heating process continued. The body is melted, or passes into a liquid state, i. e. the atoms now stand so decidedly apart from one another, that (though their general attraction keeps them together in one mass) they no longer cling tenaciously each to its neighbour, but may be moved freely among one another. In this state of a body, the power of repulsion, generated by the latent heat, seems equal to the power of attraction, which the atoms naturally exercise on one another; so that the particles appear at liberty to glide about among each other almost without friction. Moreover, in this state the particles do not admit of being compressed closer together without the exertion of a very powerful force, so powerful, that liquids were formerly supposed incompressible, till the improvements in mechanical knowledge invented more powerful compressive forces. We will now suppose the heating process resumed, so that more and more heat is infused into the liquid body: the principle of repulsion becomes stronger and stronger, and more and more overbalances the power of mutual attraction among the particles. Thus the particles recede further and further from each other, and by this increase of distance from one another, their natural power of mutual attraction, depending on their mutual distance, must

be rapidly decreasing, and consequently the antagonist principle of Repulsion will be taking more and more powerful effect in expanding the body; i. e. causing the atoms to recede further and further from each other. How fully is this view substantiated in the conversion of water into steam! One pint of water, driven off as steam from the boiler of a low pressure steam engine, fills a space of nearly 2000 pints, and raises the piston through this with a force of many thousand pounds. First, then, as the fluid on becoming steam occupies a space 2000 times greater than when in the state of water, how rapidly and widely must the atoms have receded from one another! Again, with what force must this expansion have taken place, such force being estimated by the powerful action exerted in driving up the piston! And, lastly, how wonderful is it to see this large and powerful volume of steam robbed instantaneously, by a simple cooling process, of its latent heat, and appearing again in the cold condensor as a pint of water! And let us not forget the fact, that though six times as much heat is required to convert a pint of water into steam, as to raise it from an ordinary temperature to that of boiling, the thermometer does not indicate a greater heat in the steam than in the boiling water, the excess of heat now existing in a latent state.

We see the converse to this process in the following instance. Our atmosphere is a compound substance, composed principally of two elementary

substances, in a gaseous state. Consequently, the atoms standing at a distance from each other will admit of compression, and must have a due proportion of latent heat interspersed. Now 100 pints of this atmospheric air may be compressed into a pint vessel, as in the chamber of an air gun; and if the pressure be much further increased, the atoms will at last collapse, and form an oily liquid. Also the latent heat, which was contained in such air, and gave it its gaseous form, is squeezed out in this operation, and becomes sensible all around.

All other material substances are probably affected in the same general manner as water, by the agency of heat, being convertible from the solid to the liquid, and thence to the gaseous or aeriform state. But different substances require very different degrees of heat to produce these effects. We are indeed not yet acquainted with the method of liquifying some solid substances: under our heating processes they are at once converted from the solid to the state of vapour and gas. As to the nature of this most powerful, most important agent, heat or caloric, some have supposed it a most subtile fluid, pervading all things, somewhat as water pervades a sponge; others account it merely a vibration among the atoms. We know it to be imponderable, we know it to be the cause of repulsion : but why it causes repulsion, we know not, any more than we can account for the attraction of gravity, cohesion, electricity, and magnetism. We see those powerful agencies in operation all around

us, we can ascertain the general laws by which
they act, and classify their phenomena.   Thus
much we are mercifully enabled to do, and thus
much is fully sufficient for all practical purposes of
our present comfort and well-being.   The faculties
to see into the intimate nature of these agencies,
are evidently denied us in the present life. As the
influence of the prevailing heat thus modifies the
form of solidity or fluidity in many bodies, we see
the same substance in a different state under the
warmer and colder latitudes of our globe.   Near
the equator, butter liquifies or becomes oil in the
day ; tallow candles cannot be used, and common
sealing wax will not retain impressions.   Near our
pole, quicksilver in winter is solid metal, and oils
are solid.   Philosophers have also amused them-
selves with calculating the state of fluidity or solidity,.
in which many substances would be found on the
Planets nearer to the Sun, or more remote from
him than our Earth, estimating the heat from the
Sun to vary, as other influences emanating from
a centre, are found to vary, inversely as the square
of the distance.   But we do not know what in-
fluence the atmosphere surrounding these planets
may have in modifying the heat of the Sun; where-
fore there may be far less difference of temperature
in the different Planets, than we should at first
sight suppose.   .
   As to the other powerful agent, Attraction, it
goes under different names, as it is found acting
under different circumstances.   It is called Gravi-

tation, when acting at sensible distances. Thus our own bodies and all bodies on or near the surface of the earth, are said to gravitate to the earth; in other words, they are attracted or drawn towards the centre of the earth. They are also attracted towards each other: but the quantity of matter in the earth being very much greater than the quantity of matter in the bodies on the earth, their mutual attraction is so merged in the more powerful attraction of the earth, as not to be perceptible except under particular circumstances. Weight is this tendency of a body to move towards the centre of the earth, compared with a standard, the known tendency of some other body; and as every atom is drawn towards the earth, the effort of the body to descend, or the force with which it presses down, is greater in proportion to the greater number of atoms in the body; and thus weight is an index of the comparative quantity of matter in bodies. The genius of Newton discovered, that the very same force (attraction of the earth) which is the cause of weight, and which makes the apples on a tree, when their stalk breaks, descend in a line directed to the centre of the earth, is also incessantly pulling the moon towards the earth: and moreover, that a similar force, the attraction of that mighty central body, the sun, is constantly pulling our earth and all the planets with their satellites towards the sun; and that in all cases, the intensity of this force varies inversely, as the square of the distance of the attracted body

from the centre of the great attracting body. Are
we not then on this principle wrong in asserting,
that the attraction of the earth acts with equal
force on a body, whether it be two feet, ten feet,
or fifty feet above the surface of the earth? There
is certainly a difference in the intensity of the force
in these several cases, but the difference is so very
small, as to produce no sensible effect; for the
proportion is, as square of 4000 miles (distance of
earth's centre from the surface) + 2 feet to square
of 4000 miles + 20 or 50 feet; and 50 or 100
feet would be found an infinitely small part of 4000
miles.

Whether this general attraction of matter to
matter, varying inversely as square of the distance
of the bodies, be identical with that attraction of
Cohesion which binds together the particles of the
same body, is doubted. Evidently this attraction
of Cohesion acts more readily, where the particles
are in closer contact. Our faculties being limited
and imperfect, not only do the ultimate principles
in the operations of nature lie beyond our ken;
but also our conceptions and our measurements
cannot keep pace with the measurements of space
and of quantity, infinitely minute, as well as infi-
nitely great, occurring in the works of the Creator.
Thus to our eye the particles of a body may seem
in close and perfect contact. But investigations
and reasonings tell us, that besides the evident
pores in a body, the atoms of the apparently solid
parts are no where in actual contact, but are re-

tained in their places by a balance between Attraction and Repulsion. Wherefore, although these very minute distances of the atoms may all of them be too small for our powers of conception and measurement, they may have many different degrees of proportion among themselves : and thus, according to the general law of gravitation, there would be great and numberless differences in the force of Cohesion. Any further enquiry into Cohesion would require chemical data, and other illustrations not compatible with the present occasion. I will observe, in conclusion, that Attraction in causing the atoms to cohere so as to form solid bodies, seems not to act equally all around each atom, but between certain sides or parts of one, and corresponding parts of the adjoining ones ; so that when atoms are allowed to cohere according to their natural tendencies, they always assume a certain arrangement and form, which we call Chrystalline. This fact has been called the Polarity of Atoms, and is probably in a great measure the cause of elasticity, brittleness, porosity, malleability, and other peculiarities. When its latent heat is sufficiently abstracted by frost, water, owing to this polarity of atoms, begins with shooting delicate needles across the surface, these thicken and interweave, and this new arrangement of atoms leaves pores or hollow intervals, so that the water, thus solidifying, becomes more bulky, than in its liquid state ; dilating with such force, as to burst the strongest vessels, and to split rocks, where it has

been retained in their crevices. In the present state of science there appear to be about fifty substances in nature, distinct from each other, and therefore called elements. The atoms of these different kinds of matter will not cohere and unite indifferently to form masses, as atoms of the same kind do; but there are singular preferences and dislikes among them, affinities, as the Chymists term it: and when atoms of two kinds do combine, the resulting compound generally loses all resemblance to either of the elements.

The history and classification of the facts connected with the combination and analysis of different substances, constitute the interesting and important science of Chymistry, which explains how these fifty elementary kinds of matter, by variously combining, form the endless diversity of bodies constituting the mass of our globe.

Having taken this view of those two great principles or agents in the material world, Attraction and Repulsion, we now come to a property, inherent in every atom of matter, called Inertia, the foundation of those simple general laws, by which the motions of bodies are regulated. The Latin word, inertia, signifying indolence or sluggishness, we use to indicate a total inability in matter to make any change in its state of rest or of motion, also an unwillingness and resistance to having any such change made in its state. This property was not ascertained till the age preceding Newton, and its discovery is the basis of that wonderful improve-

ment of modern above ancient science as to the
theory of motion. Our daily experience so clearly
informs us of the inability of lifeless matter to put
itself into motion from a state of rest, that the
most unlearned person would infer some external
force had been employed, if he saw a lifeless portion
of matter moved from a spot where he knew it was
previously remaining motionless. But our daily
experience seems at variance with the position,
that motion is as natural to matter as rest. For
when a body is put into motion, we see an evident
tendency to slacken its velocity, and come back
to a state of rest. But we have to consider, that
directly a body is put in motion, there are several
impeding causes immediately acting on it to lessen
and destroy its motion; and we find, that in pro-
portion as we can remove or lessen the agency of
these causes, the longer and the more uniformly
does the motion go on. This is so fully and satis-
factorily verified by experiment and observation,
that the inference is irresistible; if we could entirely
remove the agency of these disturbing causes, we
should see a body, when put into motion, con-
tinuing that motion perpetually and uniformly.
Thus when we throw a stone, the resistance of the
atmosphere, through which, a material medium, it
must force its way, keeps lessening its velocity, and
the attraction of the earth keeps pulling it down-
wards, and thus the stone, instead of moving uni-
formly on in the line in which you project it,
describes a curve, and descends to the earth. But

why does not the stone in this case immediately
fall to the ground, for the earth's attraction pulls
it downwards, directly it quits the support of your
hand ? The reason is, the stone endeavours to move
on in the line in which you project it, and with
the velocity which you give it, and this endeavour,
combined with the earth's pulling it downwards,
accounts for the curve described by it. Again,
when a horse unexpectedly stops, why is the rider
thrown over his head? or when a carriage suddenly
stops, why are you thrown forwards? In both these
cases our bodies have had a certain velocity in a
certain direction given them; and they will, as por-
tions of matter, endeavour to continue so moving
forwards, unless we by muscular exertion in oppo-
site direction counteract such tendency. You may
recognise the agency of this inertia of matter so
easily and fully in all the common phænomena of
motion, that I need not mention further proofs of
its existence. The first law of motion is the
simple declaration of this property, that a body, if
at rest, and left to itself, will remain at rest, or if
put into motion, and left to itself, will move on in
a straight line, and with an uniform velocity. Thus
there is in matter a total indifference as to rest or
uniform rectilinear motion: one is as natural to
it as the other; but there is a pertinacity in matter,
that it will make a visible and powerful effort to
continue its previous state of rest or motion. From
this indifference and this pertinacity the second law
of motion is deduced, that motion in a body is in

proportion to the force impressed, and in the direction of such force. As matter can neither generate motion in itself, nor destroy motion in itself, the motion caused in a body by a force must be regulated in quantity and direction by the force. And suppose two forces are acting on a body, it will have a tendency to move in a direction and with a velocity belonging to one of the forces, and this tendency will not prevent its receiving another tendency to move in a direction and with a velocity belonging to the other force ; and its real motion will be in a certain intermediate direction, and with a certain velocity; and the mathematician easily ascertains this direction and velocity, if he knows the directions and proportions of the forces. On the same principle, the motion of a body is ascertained, when acted on by any number of forces, if their directions and proportions are previously known. I am here supposing forces of the most simple kind, called impulsive, where a single blow or impulse is given to a body, and not repeated. Suppose any number of such blows or impulses in different directions given simultaneously to a body, we can ascertain the consequent direction and velocity of the body, which direction and velocity the body would retain for ever, if no disturbing causes came in. But we have constant forces acting upon bodies, i. e. forces whose action on a body is incessant. Thus suppose a stone falling freely from the top of a high tower. The attraction of the earth does not merely give it a single pull

downwards, but continues incessantly pulling it.
Each pull is equal in force to the previous pull,
and accordingly communicates an equal velocity.
Thus equal additions of velocity are communicated
to the stone each indefinitely small portion of
time.   The stone by its inertia retains all these
velocities, and acquires a fearful rapidity from
their aggregate effect in two or three moments.
The term momentum is used to signify the quan-
tity of motion in a body, i. e. the force with which
it would strike against another body, the force
with which it pulls in the direction in which it is
moving.   This momentum is found to depend on
two circumstances, the quantity of matter in a
body, and its velocity; so that if you know the
comparative quantities of matter in two bodies,
one for instance being 2lbs. and another 20lbs.
and the first is moving with velocity of 30 feet in
1″, the other with velocity of 2 feet in 1″, the
momentum of the first body is to the momentum
of the second body as 2 × 30 : 20 × 2 : 60 : 40.
Thus, in this instance, the smaller body would
strike against an obstacle, or pull in the direction
of its motion, more powerfully than the larger
body, in the ratio of 60 : 40 : 6 : 4 : 3 : 2.   Why is
the cannon ball more efficacious in battering down
a wall, than the battering ram used by the ancients?
The latter was some hundred times more heavy
than the former.   But by the explosive force of
gunpowder, a velocity is given to the cannon ball,
so very much greater than what the ancients could

give to their battering ram, that the momentum in the one case is fearfully greater than the momentum in the other case. A small stone, falling from the height of 3 or 4 feet, strikes with little force; but falling from the height of 40 or 50 feet, it would give a deadly blow. The repeated incessant impulses given to the stone through the small interval of its fall from the height of 40 or 50 feet by the earth's attraction, proportionally increase its velocity, and therefore its momentum. Having got a right understanding of momentum, we are enabled correctly also to understand the third law of motion. "Action and reaction are equal, and in opposite directions." Whatever momentum the action of one body communicates to another, this second body, by its inertia resisting this communication, takes away exactly as much momentum from the first body, i. e. it reacts with a force equal to the force acting on it, and in the opposite direction, and thus appears to give so much momentum to that body in this opposite direction. The action here spoken of means any force or power exerted by a body in giving or destroying motion, or tendency to motion, in another body. Thus, if a body impinges on another, or by some ligament pulls another, or by some inherent power of attraction, as in the case of magnet and iron, draws another body towards it, whatever momentum is communicated to the body acted on, the same momentum is impressed on the acting body, in the opposite direction. The horse draws a cart. The

weight of the cart, and the velocity with which it moves, is the amount of momentum given to cart by action of horse: the cart, as it were, pulls the horse back with equal force, consequently the power of the horse must be sufficient not merely to sustain this momentum or force with which the cart pulls it back, but also to overcome the inertia, which the matter in his own body exerts towards having such a velocity communicated to it. Imagine a horse drawing a carriage at the rate of six miles per hour, and that the carriage weighs twice as much as the horse. Suddenly the traces, reins, and shafts break: the effort which the horse was making one moment before, being continued, would now all be concentered in giving motion to his own body. A momentum double the momentum arising from the mere previous motion of his own body, is now virtually added to his onward movement, so that his momentum, and consequently his velocity, in that direction is trebled. A steam vessel is towing a heavy laden boat. The rate at which the steamer advances, and weight of boat, gives the momentum communicated to boat. Cut the tow rope, and keep up the same power of steam. The reaction or pulling back by the boat is now at an end, and the steamer moves on with an increase of velocity, proportioned to the momentum it was previously giving the boat, and therefore losing itself.

By animal power, by steam, water, wind, and mechanical contrivances, we call various forces into action: they communicate certain momenta. If

the bodies, to which any of these momenta are communicated, can move freely in the direction of these momenta, the whole of the momenta take effect in communicating motion. If the bodies can only move in a direction oblique to the direction of momenta, part only of the momenta take effect, and such part is easily calculated by a few simple Mathematical diagrams and principles, called the composition and resolution of forces. Also the momentum engendered by any agencies of ours impart less and less velocity to a body, in proportion as its quantity of matter is greater. But to gravity, or the earth's attraction, it is indifferent, whether a larger or smaller quantity of matter is submitted. Falling freely from rest, the larger and the smaller body acquire in the same time equal velocity. The attraction of the earth pervades every point of the circumambient space around its surface with equal power at equal heights. Consequently as a body is larger and larger, it occupies more and more points of this space, and thus is acted on by more and more of these radii or influences emanating from the earth. Some bodies fall slower than others, and some even float in the air, or are carried upwards. But this is owing to the specific gravity, not the quantity of matter. Drop a stone or a piece of metal into a bucket full of water: they rapidly sink down to the bottom. Drop a piece of wood or cork; gravity we know pulls them down, but they cannot force their way through the water. The atoms contained within

their bulk do not stand sufficiently close together,
to give a weight competent to overcoming the
resistance of the bulk of water, which in their
descent they would displace.   Our atmosphere is a
material medium, though very attenuated, compared
with water. But a similar principle holds. A body in
falling through air must displace the particles of
air, that are in its way downwards: these offer a
certain resistance to being displaced. Consequently
if the body be of a very spongy or porous texture,
the weight or tendency downward of a certain bulk
of such body, removing an equal bulk of air, may with
difficulty overcome the resistance of that quantum
of air, so as to descend slowly: or may be unequal
to overcoming it, and thus may float about.   Even
a dense body, like gold, may be beat out into so thin
a plate, that its weight shall not be competent to
overcome the resistance of the large surface of air
opposing its descent. Our view of momentum and
of action being equal to reaction, enables us to see
the leading principle of the several mechanical con-
trivances for giving facility to the raising of weights
and overcoming resistances. The power, which we
bring into action, may, unaided by such contrivances,
be equal to only one-tenth of the resistance to be
overcome ; but if by use of machinery I can so
connect the action of the power with the action of
the resistance, that the momentum of the one shall
act directly against the momentum of the other, and
if at the same time I can give the power a velocity
twelve times greater than the velocity of resistance,

the momentum of power becomes greater than the momentum of resistance, and overcomes it. Thus a bale of goods is to be raised from the ground into a warehouse thirty feet above the ground. The crane, a wheel and axle, causes the power of the man's hand turning round the wynch to act directly against the pressure downwards of the weight, as the turning round of the wynch makes the axle revolve, and thus the rope attached to the weight coils round the axle, and pulls up the weight. Moreover, while the hand travels once round the circumference described by the wynch, the weight is drawn up through space equal to the circumference of axle. These two circumferences, therefore, give the ratio of the velocities. And thus, by making wynch larger, and axle smaller, you may give your power a greater and greater relative velocity. Where the lever, or pullies, or inclined plane, or wedge, or screw, are used to give a mechanical advantage, it is all on the same principle, to cause by their intervention the momentum of the power to act directly against the momentum of resistance, and by its greater relative velocity to increase in the required degree the momentum of power; so that having by its action imparted to the resisting substance a momentum equal and opposite to the momentum to be overcome, the power may still have some excess of momentum to put itself into actual motion. Great ingenuity has been shewn in the construction of such machinery, and the experience of successive ages has suggested most

useful modifications and combinations. The action
of the power must be made to take effect in a
direction opposite to the resistance; otherwise its
entire momentum does not take effect, only part
of it.   Again, as your mechanical advantage is
greater, the difference of the relative velocities
being greater, the progress of the effect is slower.
Therefore, where rapidity of effect is necessary, it
must be accomplished by giving up such mecha-
nical advantage, and by due increase of absolute
power.

   A few words, in conclusion, on Centre of Gravity
and Centrifugal Force.   Suppose a point within a
solid body to be resting on a support, and the body
to have free liberty of inclining or leaning downwards
on any side of this support, if its weight pressed
down more powerfully on any one side of this
support than on the opposite side: now if this
point were so situated within the solid, that the
momenta arising from the weights of the several
particles, on *any* one side, were equal to the
momenta arising from the weights of the particles
on the opposite side, the point is called the centre
of gravity, i. e. the tendency downwards arising
from the gravity of the particles in any one
direction from that centre, is exactly equal to the
tendency downwards arising from the gravity of
the body's particles in the opposite direction.
And this point being supported, these downward
momenta on the opposite sides of this support
thus act against one another, and counteracting

one another, keep the body at rest, if no other force but gravity be acting on it. We must remember, that the momenta with which the weight of particles act in pressing down the body on any side of this supported point, depend not merely on their number, but on their relative distances from the perpendicular line passing through this centre, as necessarily follows from the properties of the lever. Allowing all this, we see, if the centre of its gravity be supported, the weight of the body will not give it a tendency to fall on any side of such support. But if the support be sustaining some other point to the right or left of the centre of gravity, the downward momenta of particles on some one side of such point will be more powerful than the downward momenta of the opposite side, and thus the body will have a tendency to fall on the former side of such point. Many common phænomena in bodies around us, and the motions of our own bodies, furnish interesting illustrations. If a wall inclines, so that the perpendicular line from its centre of gravity no longer falls within the base of the wall, the weight of the upper half of the wall is only counteracted by the cementing power of the mortar holding the bricks or stones together. As we stand, the lines drawn from the toe and heel of one foot to the toe and heel of the other foot give the base or area of our support. Place a large bundle on a man's back ; his centre of gravity is now so altered, that he instinctively leans forward, to enable the perpendicular from

this centre to fall within the support of his feet.
Place the bundle in his arms before him, he leans
backwards; place it on his head, he stands upright,
on the same principle. For the same reason, you
throw your body forwards in rising from your
chair. And the skill of the ropedancer is shewn, in
keeping this perpendicular within the very narrow
base of his support. His long pole, loaded at
one end with lead, enables him instantaneously,
on the principle of the lever, to shift the position
of his centre of gravity. In all cases, the broader
the support on which a body rests, and the lower
down the centre of gravity, the more difficult is it
to throw down such a body, because there are
more impediments in getting the perpendicular
from its centre of gravity beyond the base of its
support.

When any particles of matter are moving in a
curve, we see a constant effort exhibited in them
to fly off from this curvilinear direction, hence
called centrifugal force; a tendency to fly off from
the centre of the circle in which we suppose them
moving. This centrifugal tendency is the effect
of the general inertia of matter, inclining any
portion of matter to persevere straightforwards,
without deviation, in the direction in which it is
moving. A straight line is the natural direction
in which matter moves, if put into motion, and
left to itself. Wherefore, if we see it moving in a
curve line, we infer some constant force is acting
on it, and incessantly pulling it out of its recti-

linear direction. A curve may be considered as made up of a number of very little straight lines, each line constantly deviating from the direction of the previous line. The body moving in this previous line endeavours to persevere in it; therefore, on being pulled out of this line into the next very small line, it exhibits a sensible resistance to being thus forced out of its previous direction; but, when pulled into a new direction, it endeavours to continue in this new direction, and immediately again exerts resistance to being made to deviate into another new direction. These directions are the tangents to the curve at the points where they end; and thus we say a body has a perpetual tendency to fly off in a tangent. Suppose a stone whirling round and round in a sling; its movement is confined and constrained to a circle by the string; but at every point of that circle it is exerting an effort to fly off in the direction of a tangent to that point; and this effort increases as velocity increases, because in that case its momentum increases; and this tendency to persevere at any point in a rectilineal direction keeps the string powerfully stretched, even at the upper part of the circle, where the weight of the stone, if not counteracted by a stronger force, would cause the string to collapse.

A loaded coach, whirling rapidly round a corner, upsets: the wheels are dragged round the corner; but the body of the coach, moving with great momentum from the rapidity, shoots powerfully on in

its previous direction; and the overturning is made
still more certain, if the height of the centre of
gravity, from luggage being piled on the top,
lessens the stability of the carriage on its base.
In an equestrian circus, why does the horse,
galloping round in a circle, incline his body in-
wards towards the centre of the circle, and why
does the man, standing on his back, do the same ?
Both bodies, by the general inertia of matter, have
a powerful tendency to move straightforwards, at
every point of the circle, in the direction of a
tangent ; and this centrifugal tendency makes
each body liable to slip down or fall towards the
outside of the circle, and this liability is counter-
acted by each body inclining inwards. Also, if
the horseman finds his body falling inwards, he
quickens the horse's pace ; this increases the
momentum of his own body's motion, and thereby
its tendency to fall outwards ; but he slackens the
pace, and thereby lessens this tendency, if he per-
ceive himself falling outwards. Again, why does
the earth bulge out at the equator and flatten at
the poles ? and why does the weight of a body
lessen, if carried from a higher latitude towards
the equator ? Because the revolving of the earth
round its axis causes the surface of the earth, and
all things on that surface, to revolve with greater
rapidity at the equator than at higher latitudes ;
thus their centrifugal force, or tendency to fly
outwards, is increased, and this must lessen the
effect of gravity, or the earth's attraction. Lastly,

when the astronomer looks forth into the heavens, he perceives the planets nearer the sun moving in their orbits with greater velocity than the more remote planets. Why is this? The great Creator has placed the immense body of the sun, as the fountain of light and heat, in the common focus of all their elliptical orbits; and has ordered the attractive influence of this vast central body to draw the planets towards his centre with a force that increases in the inverse duplicate ratio of their distances. The Almighty hand, we will suppose, then launched a more distant planet in such direction, and with such velocity, that its tendency to maintain a rectilinear motion sufficed to prevent the sun's attraction from drawing the planet directly downwards, and yet was not so great as to enable the planet to continue too far receding from the sun, but was so balanced and limited, that the planet, incessantly drawn towards the sun, and incessantly striving to maintain a rectilinear motion, revolves from age to age in its orbit round the sun. But when we come to a nearer planet, how are the centripetal and centrifugal forces to be thus duly balanced, unless the planet here be launched with a greater velocity? Suppose the distance from the sun be now only half of the distance in the former instance, the attractive influence has increased fourfold. Wherefore this planet must be projected with a greater velocity, that its centrifugal force, i. e. its endeavour to maintain a rectilinear motion, may sufficiently coun-

teract the great increase of the centripetal force.
The velocity, however, is increased in a far smaller
ratio than the centripetal force, because the planet,
now moving in a smaller orbit, is more abruptly
and more incessantly pulled out of its rectilinear
direction, and therefore its inertia, having more
violence done it, reacts more powerfully, and
exerts a more powerful centrifugal force.

I have thus endeavoured to set before you the
wonderful operation of heat, in modifying the form
and external constitution of all material substances,
and its influence on those two great agents of
nature, Attraction and Repulsion. Secondly, I have
explained that inertia, inherent in every atom of
matter, out of which grow the three simple general
laws, that regulate the phænomena of motion
occurring around us. I conclude with earnestly
recommending a careful perusal of " Wood's
Mechanics" to the student, for a more full and
accurate information on the composition and
resolution of motion, the mechanical powers, the
collision of bodies, and the rectilinear motions of
bodies accelerated or retarded by uniform forces.

I append a short statement of Infinitesimals.

INFINITESIMALS.

Quantities may be regarded as finite in their magnitude, or as infinitely small, or as infinitely great. But these three terms are to be understood in a relative, not in an absolute, sense. For the same quantity will be considered as finite, or as infinitely small, or as infinitely great, according to the relative magnitudes of the quantities with which it is brought into comparison. And this view will give different orders of Infinitesimals, as a quantity may be proved infinitely small, compared with a quantity which had been previously proved infinitely small, compared with a quantity termed finite.

Thus, let x be Infinitesimal of 1st order, i. e. infinitely small compared with a finite quantity 1; then $1 : x :: x : x^2 \therefore x^2 =$ Infinitesimal of 2d order; and $x : x^2 :: x^2 : x^3 \therefore x^3 =$ Infinitesimal of 3d order; and so on. Let x and y be Infinitesimals of 1st order. Then $1 : x :: y : z \therefore z =$ Infinitesimal of 2d order. But $xy = z \therefore$ Product of 2 Infinitesimals of 1st order gives Infinitesimal of 2d order; and, by analogy, Rectangle under two evanescent lines, Infinitesimals of 1st order is Infinitesimal of 2d order.

Let $a =$ a finite quantity. Then $1 : x :: a : ax$ $\therefore ax$, or Product, or Rectangle, under a finite magnitude, and one which is Infinitesimal of 1st order $=$ Infinitesimal of 1st order. Let m be a

quantity infinitely great; take $1 : x : : m : a$.˙.
$mx = a$, i. e. a quantity infinitely small, multiplied
by a quantity infinitely great, gives a result $=$ finite
quantity: and this holds in all analogous processes.
Then in Lemma 2, the Parallelograms, which are
Infinitesimals of 1st order, being infinite in number,
make up by their aggregate sum, a finite mag-
nitude.

MATHEMATICAL PRINCIPLES

OF

NATURAL PHILOSOPHY.

SECTION I.

OF THE METHOD OF PRIME AND ULTIMATE RATIOS, BY THE
HELP OF WHICH THE FOLLOWING PROPOSITIONS ARE
DEMONSTRATED.

LEMMA I.

*Quantities, and the ratios of quantities, which, in any finite time,
tend continually to equality; and, before the end of that time,
approach nearer to each other than by any given difference,
become ultimately equal.*

IF you deny it, let them be ultimately unequal;
and let their ultimate difference be D. Therefore
they cannot approach nearer to equality than by
that given difference D. Which is against the
supposition.

Of any two quantities or two ratios coming
under this Lemma, both may, and one must, be
variable: as a change is supposed to be going on,

by which the difference between them is con-
tinually decreasing, so that in a finite time this
difference becomes, in one case, a quantity in-
finitely small, compared with either of the two
quantities; in the .other case, a ratio infinitely
small, compared with either of the two ratios.

Suppose a triangle, A B C, (B a right angle,) the
base A B divided
into three equal
segments by the
ordinates, D d,
and Ee: suppose
the lines A D,
D d, completed
into rectangle
D M, the lines D E, E e, into rectangle E N, and the
lines E B, B C, into the rectangle B P. The dif-
ference between the area of the triangle A B C and
the area comprised in the three rectangles is equal
to half the last rectangle B P. Now suppose the
segments of the
base A B to be
each bisected,
so that the base
A B is now di-
vided into six
equal segments;
suppose corre-
sponding ordinates to be drawn to each segment,
and these ordinates with their segments to be
completed, as before, into six rectangles. The
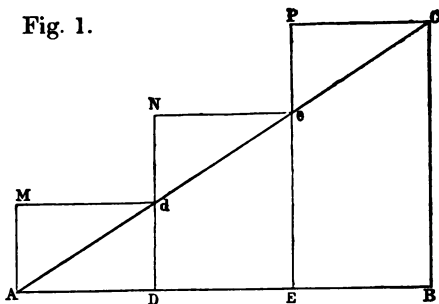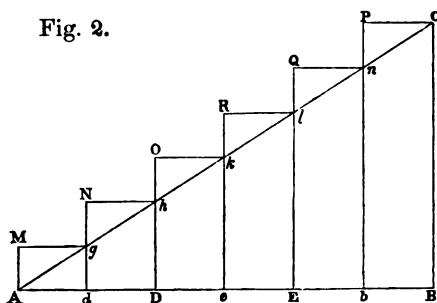
difference between the area of the triangle A B C, and the area of the sum of the rectangles, is now equal to half the last rectangle B P, therefore equal to half the difference in the preceding case. Suppose this process to be continued of bisecting the segments, drawing the ordinates, and completing the rectangles: each step of this process reduces one half the difference between the area of the triangle and the sum of the rectangles: and this difference at length becomes equal to half of a rectangle, having the given finite line C B for its altitude, but for its base a line which has dwindled down to a mere point, and therefore infinitely small, considered as a right line, and compared with a right line of a finite length. Therefore, as rectangles of equal altitudes are as their bases, the rectangle, whose half always expresses the difference between the two areas supposed in the hypothesis, becoming smaller and smaller, is at last an area less than the area of any assignable rectangle, is an area infinitely small, compared with the area of either of the two original areas, which have by the supposed process been approximating to an equality. This rectangle, in short, becoming equivalent to the product of a finite number, multiplied into a number infinitely small, has become equivalent to an infinitesimal of the first order.

We should consider the character or species of the two quantities approximating to equality, whether they be lines, areas, or solids; the vanishing difference between them must be ex-

pressed by a quantity of the same kind or charac-
ter, and must be proved to become within a finite
time infinitely small compared with either of
the two quantities. Considered by itself, it may
ultimately be *o,* or an infinitesimal, or a finite
magnitude, according to the circumstances of the
particular case.

The limit of a quantity, or its ultimate value,
expresses the state to which the quantity is con-
stantly approximating by the supposed process,
which state it never absolutely reaches; but ap-
proaches so near to it, that no error arises in
conclusions drawn from assuming the quantity to
have ultimately reached that state.

Thus coincidence with the triangle A B C is the
limit, or ultimate state, of the sum of the circum-
scribed rectangles.

$\frac{ax + b}{cx + d}$ This ratio has two limits, $\frac{b}{d}$ if x decreases
and vanishes, for then ax and cx are each ultimately
$= o.$

And $\frac{a}{c}$ if x increases and becomes infinite, for

divide by x and the ratio $= \dfrac{a + \dfrac{b}{x}}{c + \dfrac{d}{x}}$

And $\frac{b}{x}$ and $\frac{d}{x}$ are each ultimately $= o.$

The second condition specified in this Lemma,
that " the quantities before the end of finite time
approach nearer to each other than by any given

difference," excludes all the possible cases, intimated in Thorp's Commentary of Quantities, perpetually approaching to equality, yet never approaching nearer than by a given difference.

. LEMMA II.

If in any figure A a c E, *terminated by the right lines* A a, A E, *and the curve* a c E, *there are inscribed any number of parallelograms* A b, B c, C d, *&c. contained under equal bases* A B, B C, C D, *&c. and the sides* B b, C c, D d, *&c. parallel to* A a, *the side of the figure; and the parallelograms* a K b l, b L c m, c M d n, *&c. are completed. Then, if the breadth of those parallelograms is diminished, and their number is augmented continually; I say, that the ultimate ratios, which the inscribed figure* A K b L c M d D, *the circumscribed figure* A a l b m c n d o E, *and the curvilinear figure* A a b c d E, *have to each other, are ratios of equality.* (Fig. 3.)

For the difference of the inscribed and circumscribed figure is the sum of the parallelograms K *l*, L *m*, M *n*, D *o*, that is, (because of the equality of all their bases,) the rectangle under one of their bases K *b*, and the sum of their altitudes A *a*; that is, the rectangle A B *l a*. But this rectangle, because its breadth A B is diminished indefinitely, becomes less than any given rectangle. Therefore (by Lemma I.) the inscribed and circumscribed, and much more the intermediate curvilinear figure

Fig. 3.
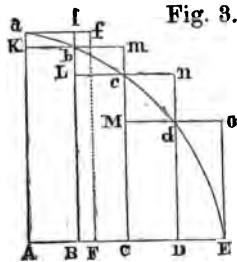
become ultimately equal. Which was to be demonstrated.

•

The inscribed and circumscribed parallelograms are ultimately infinitesimals of the first order, as their altitudes are always finite lines, proportionate to the ordinates of the curves; and their bases are ultimately evanescent lines, mere points; their number is infinite. The little parallelograms, which are their differences, are ultimately infinitesimals of the second order, as both their altitudes and bases are ultimately evanescent lines. Thus each of these differential parallelograms is infinitely small, compared with the inscribed or circumscribed parallelograms, of which it is the difference. The curvilinear area is always the intermediate quantity between, and is the limit of the sums of the inscribed and circumscribed parallelograms; and take any inscribed with its corresponding circumscribed parallelogram, equality is the limit of their varying ratio.

LEMMA III.

The same ultimate ratios are also ratios of equality, when the breadths A B, B C, C D, *&c. of the parallelograms are unequal, and are all diminished indefinitely.* (Fig. 3.)

For let A F be equal to the greatest breadth; and let the parallelogram F A *a f* be completed. This will be greater than the difference of the inscribed and circumscribed figures; but, because

its breadth A F is diminished indefinitely, it will become less than any given rectangle. Which was to be demonstrated.

Cor. 1. Hence the ultimate sum of the evanescent parallelograms coincides in every part with the curvilinear figure.

Cor. 2. Much more does the rectilinear figure, which is comprehended under the chords of the evanescent arcs *a b, b c, c d,* &c. ultimately coincide with the curvilinear figure.

Cor. 3. As also the circumscribed rectilinear figure, which is comprehended under the tangents of the same arcs.

Cor. 4. And, therefore, these ultimate figures (as to their perimeters *a c* E) are not rectilinear, but curvilinear limits of rectilinear figures.

OBSERVATIONS.

Cor. 1. The coinciding here mentioned does not mean that the parallelograms ever actually coincide with the curvilinear figure; but that their limits do, a state they never actually reach, but so nearly reach, that no error arises from assuming the ultimate coincidence.

Cor. 2. and 3. The figures here mentioned are always intermediate quantities between the sums of the parallelograms and the curvilinear area.

Cor. 4. The curve, *a* E, must be regarded as the limit of the perimeter formed by the chords, or by the tangents.

We have seen how by taking their limits or ultimate ratios, we may suppose certain rectilinear figures to become ultimately commensurate with curvilinear areas; and this will enable us, by a much shorter process than the old method of exhaustions, to ascertain with sufficient accuracy the ratios and dimensions of curvilinear spaces. Thus, if two curvilinear figures will admit of a series of parallelograms to be placed in the one, and a corresponding series to be placed in the other, as supposed in

LEMMA IV.

If in two figures, A a c E, P p r T, *there are inscribed (as before) two series of parallelograms, an equal number in each; and, their breadths being diminished indefinitely, if the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same; I say, that those two figures* A a c E, P p r T, *are to each other in that same ratio.*
(Fig. 4.)

For, as the parallelograms in one are severally to the paral-lelograms in the other; so, by composi-tion, is the sum of all in one to the sum of all in the other; and so is one figure to the other; because (by Lemma III.) the former figure is to the former sum, and the latter figure

Fig. 4.
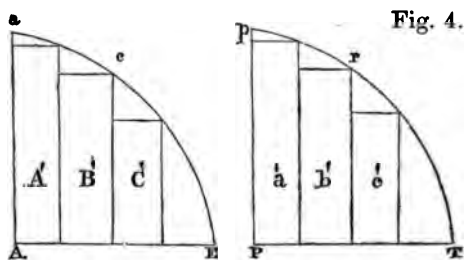
to the latter sum, in the ratio of equality. Which was to be demonstrated.

Cor. If there be two quantities of any kind, which are divided into the same number of parts, if these parts, when their number is continually increased, and the magnitude of each continually diminished, (so as to exhaust the original quantities,) are to each other in a given ratio; the whole quantities (componendo, Prop. 12. B. v. El.) will be in that ratio. We can thus compare the area of an ellipse with the area of circle on major axis, as diameter.

Newton's proof of his fourth Lemma is remarked on, as too briefly expressed, because the propositions, componendo and alternando, are proved for ratios actually the same, not ultimately the same. But the parallelograms are given as vanishing in a given ratio; therefore in their evanescent state, just before vanishing, they are so indefinitely near having that ratio, that no error will arise in conclusions built on assuming the given ratio as their ratio in their evanescent state. The additions made to Newton's proof, issue in demonstrating this.

LEMMA V.

All homologous sides of similar figures, whether curvilinear or
rectilinear, are proportional; and the areas are in the duplicate
ratio of the homologous sides.

Newton gives no proof, and none is wanted. Book vi. El. Prop. 4. 19. 20. prove of similar

rectilinear figures, that the homologous sides are proportional, and the areas in the duplicate ratio of the homologous sides. And assuming the following definition; " one curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other." We see that two similar polygons may be inscribed in two similar curvilinear figures, the sides of the polygons being chords of small arcs of the curves, that the number of these sides or chords may be indefinitely increased, the polygons remaining similar, and ultimately becoming equal to the curvilinear figures; and that the polygons being divisible into an equal number of similar triangles, the chords which are the bases of their similar triangles are proportional, and thus componendo the sum of the chords in one figure is always proportional to the corresponding sum of chords in the other figure. But these sums of the chords are ultimately equal to the perimeters of the curvilinear figures. Thus all the homologous sides of these curvilinear figures are proportional, and their areas in the duplicate ratio of these sides. Thus let A C B, *a c b*, be two similar figures, of which the sides A B, A C, B C, are homologous to *a b*, *a c*, *b c*, respectively ; then

Fig. 5.

by definition, if A D E B C be a polygon inscribed
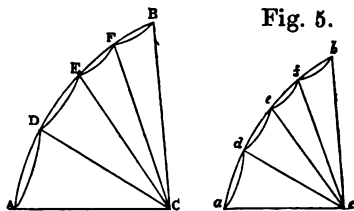
in A B C, a similar polygon $a\,de\,b\,c$ may be inscribed in $a\,b\,c$.   Join C D, C E, and $c\,d$, $c\,e$, &c. dividing the polygons into the same number of similar triangles,

$$\therefore \text{A D} : \text{A C} = a\,d : a\,c,$$

$$\text{alt}^{\text{do}} \text{A D} : a\,d = \text{A C} : a\,c,$$

Similarly $\text{D E} : d\,e = \text{DC} : d\,c = \text{A C} : ac,$

$$\text{E F} : e\,f = \text{A C} : a\,c,$$

$$\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

therefore, componendo

$$\text{A D} + \text{D E} + \text{E F} + \&c. : a\,d + de + ef + \ldots = \text{A C} : ac.$$

Now this being always true, will be true when the number of sides is increased, and their magnitudes diminished, without limit;

$$\therefore \lim. \text{AD} + \text{DE} + \text{EF} + \ldots : \lim. ad + de + ef + \ldots = \text{AC} : ac,$$

and therefore by Lemma III. Cor. 3.

$$\text{A D B} : a\,db = \text{A C} : a\,c,$$

$$= \text{B C} : b\,c.$$

Again,

polygon A D E B C : polygon $adebc = \text{A C}^2 : ac^2,$

and this being always true, will be true in the limit;

$$\therefore \text{ limit polygon A D E B C} : \text{limit } adebc = \text{AC}^2 : ac^2;$$

therefore by Lemma III. Cor. 1.

curvilinear fig. ABC : curvil. fig. $abc = \text{AC}^2 : ac^2$

$$= \overline{\text{ADB}}|^2 : \overline{adb}|^2$$

$$= \text{BC}^2 : bc^2.$$

Cor. If A C B, $ac\,b$, be two similar figures, and C E, $c\,e$, be equally inclined to A C, ac, then $\text{A C} : \text{C E} = ac : ce$.   Hence also this definition,

Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such, that if any two other distances be drawn equally inclined to the former, the four are proportional.

If the definition of similar curvilinear figures given in this Cor. (which is deduced as a proprium from the former definition) be assumed as the differentia or distinguishing property, then the former definition follows as a Cor. or proprium from it.

In Lemmas VII. VIII. IX. we have occasion to elongate a chord, subtending a certain arc, and on the lengthened chord to draw an arc similar to this arc. The following problem gives the method of doing this.

Prob. Let the chord A B of the curve A C B be produced to *b,* to describe on A *b* a curve similar to A C B.

In A C B take any point P, join A P, and produce A P to *p,* so that $Ap : Ab = AP : AB$; then if the curve A *p b* be the locus of all points, whose position is determined in the same manner as that of *p,* it will be similar to the curve A P B.

Fig. 6.
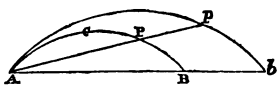
Lines are supposed to be traced out by motion of a generating point. If the change of direction is incessant, a curve line is traced out.

Def. 1. The *tangent* to a curve A B at A is the straight line, in which the generating point would move, if instead of changing the direction of its motion, it moved on in the direction which it had at A.

Def. 2. The curvature of a curve is said to be *continued,* when the curve is wholly convex or concave to a given straight line on the same side of it, and when the change of direction is not abrupt, but gradual; that is, if A T U, B T, (Fig. 7. Lemma VI.) be tangents at A and B, in a curve of continued curvature, the angle B T U as B moves up to A, diminishes through every change of magnitude from its original value, and ultimately vanishes.

The incessant and gradual change of direction in the motion of a generating point, will cause the angle contained by tangent and curve to be less than any rectilinear angle.

LEMMA VI.

If A C B *be an arc of continued curvature,* A B *the chord, and* A T U *the tangent at* A, *the angle at* B A T *between the chord and tangent, as* B *moves along the curve towards* A, *and ultimately coincides with that point, continually diminishes and ultimately vanishes.*

Let the tangents at A and B meet in the point T; then the angle B T U measures the change in

E 2

the direction of the motion of the generating point
which takes place in passing
from B to A, and since the
curvature is continued, this
angle, as B moves towards and
ultimately coincides with A,
continually diminishes and ulti-
mately vanishes, therefore *a
fortiori* the interior angle B A T continually di-
minishes and ultimately vanishes.

Fig. 7.
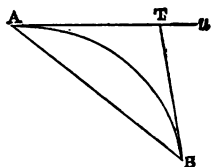
In illustration of this conclusion, suppose a
circle to osculate the curve at A, and to pass
through the point B, so as to have A T for its
tangent, and A B for its chord. The angle be-
tween chord and tangent is always equal to the
angle in the alternate segment of the circle, i. e.
the segment of the circle standing on the diminish-
ing chord A B. This angle continually lessens as
B A lessens, and when B A becomes less than any
assignable quantity, becomes less than any assign-
able angle. (See Figure 18, where P R and P *q*
answers to A T and A B.) As the angle between
chord and tangent ultimately vanishes, chord and
tangent ultimately coincide ; hence the following
Corollary.

Cor. Similar conterminous arcs, which have
their chords coincident, have a common tangent.

Let the similar conterminous arcs A P B, *a p b*
have their chords A B, A *b* coincident, and let

A P p, A Q q be any other coincident chords;
then since the curves

Fig. 8.

are similar **A P** : **A** p
= **A B** : **A** b = **A Q** :
A q, therefore the arcs
A P, **A** p are similar, that
is, the chords of the
similar arcs **A** P, **A** p co-
incide.   Now let P and p move up to A, the arcs
A P, **A** p, since they are always similar, will vanish
together, and **A** P p in its ultimate position will be
a tangent to each, that is, the arcs **A** B, **A** b have
a common tangent.
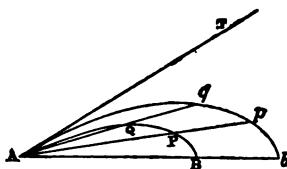
Def. The *subtense* of an arc is a straight line,
drawn from one extremity of the arc to meet at a
finite angle the tangent to the arc at its other
extremity.

LEMMA VII.

If B D *be a subtense of the arc* A C B *of continued curvature, the*
  chord A B, *the arc* A C B, *and the tangent* A D, *when* B D
  moves parallel to itself up to A, *are ultimately equal to each other.*

Produce **A** D to any fixed point d, and draw $d b$
parallel to D B to

Fig. 9.

meet **A** B pro-
duced in b ; on **A** b
describe the arc
A $c b$ similar to
A C B, and as B
moves up to **A**, let
A $c b$ so alter its form as to be always similar to
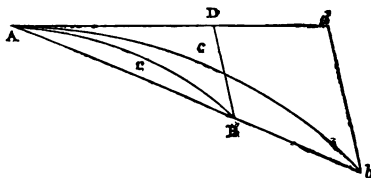
A C B; hence the two arcs have a common tangent, and the three lines A B, A C B, A D are always proportional to A *b*, A *c b*, A *d*. Now as B moves up to A, the angle *b* A *d* continually diminishes and ultimately vanishes, (Lemma VI.) the point *b* moves up to and coincides with *d*, and therefore A *b* and A *d*, and therefore, *a fortiori*, the intermediate arc A *c b*, are ultimately equal. Hence A B, A C B, A D, which are always proportional to them, are ultimately equal to each other.

The point *d* remains fixed, and the line *d b* remains fixed and invariable. As B moves up towards A, the chord A B moves up, of course, towards A D, round A as the centre, and its elongation A *b′* meets *b d* higher and higher up towards *d*, as B approaches nearer and nearer to A ; and on this

Fig. 10.
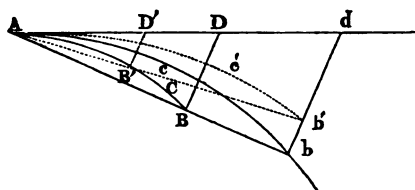
line A *b′* of finite length, in its different positions, is always described an arc A *c′ b′*, similar to the original decreasing arc A C B. Supposing this original arc to be a circular arc, and in its original position to have been an arc of 30°. The similar arc A *c b*, on the elongated chord, will also be a circular arc of 30°, and therefore belonging to a larger circle than the circle to which the arc A C B belongs. When B has moved so much

nearer to A, that the arc A B′ would now only measure 15°, the similar arc A $c′$ $b′$ will measure 15°, and must have unbent itself, i. e. have less curvature, and become the arc of a larger circle. This process goes on; the finite arc A $c′$ $b′$ keeps unbending itself, i. e. having a curvature less and less, approximating more and more to a rectilinear direction.

In the earlier Lemmas, Newton proves the ultimate equality of quantities by shewing that their difference is a quantity ultimately equivalent to nought, when compared with the quantities in question. In this and the two subsequent Lemmas, he has to prove that certain quantities will vanish in a certain ratio; i. e. he has to prove the ratio which they are indefinitely near attaining in their evanescent state, in that state in which they seem to be beyond our cognizance or examination. Admirable, therefore, is the skill of Newton in constructing a demonstration, which gives certain finite quantities, always proportional to the quantities which are decreasing into an evanescent state; and these finite quantities consequently give the ratio with which the other quantities vanish.

Cor. 1. Since the proof holds whatever be the inclination of B D to the tangent, provided it be finite, if B E be a subtense

Fig. 11.

making any other finite angle with A D, the tan-

gents A E, A D, and the chord and arc are ultimately equal.
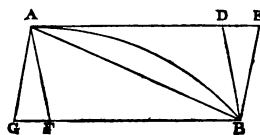
Cor. 2. Also if the parallelograms A D B F, A E B G be completed, since A D, A E are always equal to B F, B G respectively, the lines A D, A E, B F, B G are ultimately equal to the chord and arc ; and in all geometrical investigations the ultimate values of all these lines may be used indiscriminately for each other.

The segments D E, G F of the abscissas A D, A E, B F, B G, must be infinitesimals of the second order, as they are infinitely small, or = *o*, when referred to these abscissas, which are infinitesimals.

This Lemma, if the curvature be given finite at A, might be proved from Fig. 18, P V being drawn parallel to subtense R *q*; as in that case from similar triangles, P*q* : P R :: P V : *q*V. But when P *q* becomes less than any assignable quantity, the finite lines P V, V *q* approach nearer to the ratio of equality, than by any assignable difference.

LEMMA VIII.

If the straight lines A R, D B R, *which meet in* R, *make with the chord* A B, *the arc* A C B, *and the tangent* A D, *the triangles* A B R, A C B R, A D R; *these three triangles, when* B *moves up to* A, *are ultimately similar and equal to each other.*

Produce A D to a fixed point *d*, and draw *d b r* parallel to D B R, meeting A B, A R produced in

$b, r.$ On A b describe the arc A c b similar to A C B,
and let it so alter
its form, as B moves
up to A, as to be
always similar to
A C B.   Then the
two arcs will have
a common tangent
A D d, and the three
triangles   A B R,
A C B R,   A D R
will be always similar to the three A b r, A c b r,
A d r respectively, and will bear each to each
the same ratio, viz. that of R A^2 : r A^2; hence,
alternando, A B R : A C B R : A D R = A b r :
A c b r : A d r.

Fig. 12.

Now let B D move parallel to itself up to A, then
the angle b A d continually diminishes and ulti-
mately vanishes ; and A b, and therefore the inter-
mediate arc A c b, ultimately coincide with A d ;
hence the triangles A b r, A c b r, are ultimately
similar and equal to A d r ; therefore the triangles
A B R, A C B R, A D R, which are always pro-
portional to them, are ultimately similar and equal
to each other.

In saying angle b A d ultimately vanishes, the
meaning is, angle b A d may be taken less than any
assignable angle, and thus the limit of the varying
ratios of the finite triangles r A b, r A c b, r A d, is
the ratio of equality.
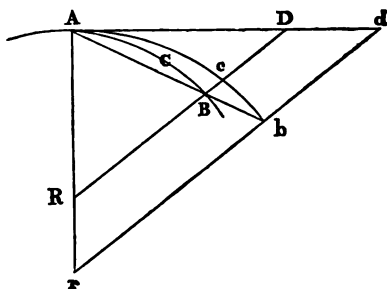
Obs. In the Lemma R B D is supposed to move parallel to itself towards A, that is, *b* moves along *r d* fixed, and the triangles A *b r,* A *c b r,* A *d r* are always finite ; but the same thing will be true, if R B D revolve round R fixed, in which case also, though *r* moves off to an infinite distance, and the triangles A *b r,* A *c b r,* A *d r* increases indefinitely, they will be ultimately similar and equal to each other.

Newton did not contemplate R B D revolving round R fixed, his words being, " triangula tria, semper finita, *r* A *b,* *r* A *c b,* *r* A *d.*"

LEMMA IX.

If the right line A E *and the arc* A B C, *given in position, cut each other in a finite angle at* A, *and the ordinates* B D, C E *be drawn, making any other given angle with* A E; *when* B D, C E *move parallel to themselves up to* A, *the limiting ratio of area* A B D : *area* A C E *equals that of* A D^2 : A E^2.

Produce A E to a fixed point *e,* and take A *d* in A *e* such, that A *d* : A *e* = A D : A E. Draw *d b, e c* parallel to DB, or EC, meeting A B, A C produced in *b, c* ; and on A *c* describe an arc similar to A B C : this arc shall pass through *b,* for by similar triangles and by construction,
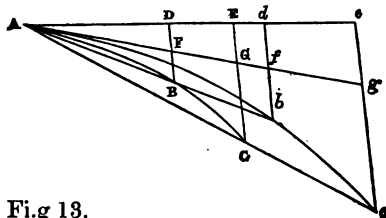
Fi.g 13.

A B : A *b* = A D : A *d* = A E : A *e* = A C : A *c,*

and therefore (Cor. Lemma V.) *b* is a point in the

arc. As B and C move up to A, let the curve Abc so alter its form as to be always similar to A B C, then the area A B D will be always similar to Abd, and A C E to Ace. Hence

$$\text{area ABD : area A}bd = \text{AD}^2 : \text{A}d^2 = \text{AE}^2 : \text{A}e^2$$
$$= \text{area ACE : area A}ce,$$
$$\therefore \text{ area ABD : area ACE} = \text{area A}bd : \text{area A}ce.$$

Also the two arcs being similar have a common tangent at A, let this be A F Gfg; and let B D, C E move parallel to themselves up to A; then the angle cAg continually diminishes, and ultimately vanishes, and therefore

$$\text{L. R.}^* \text{ area A}bd : \text{area A}ce = \text{L. R. } \triangle \text{ A}fd : \triangle \text{ A}ge$$
$$= \text{L. R. A}d^2 : \text{A}e^2$$

Hence

$$\text{L. R. area ABD : area ACE} = \text{L. R. area A}bd : \text{area A}ce$$
$$= \text{L. R. A}d^2 : \text{A}e^2$$
$$= \text{L. R. AD}^2 : \text{AE}^2.$$

A E is the axis, by motion along which the ordinate D E traces out the curve A B C. As the angle is given finite, at which the axis meets the curve at A, it is not a tangent to the curve. The curve is convex or concave towards the axis, according as the ordinate increases, faster or slower, (in greater or lesser ratio,) than the corresponding abscissas of the axis. If the ordinate varies as the abscissa, it traces out a straight line, and the area

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* L. R. signifies " limit of the ratio," or " limiting ratio."

always varies as the square of the abscissa. This
ratio is proved to be the ultimate ratio of even the
curvilinear triangles A B D, A C E, i. e. the ratio
with which they vanish, supposing the points C c B
to move backwards and fall in upon A. This
ultimate ratio of the evanescent areas is identical
with the prime ratio, i. e. the ratio of the nascent
areas, the ratio which they would be indefinitely
near having at the moment of their generation,
ipso motûs initio of the generating ordinate. The
evanescent or nascent curve so very nearly co-
incides with the tangent, that the ordinate to the
curve seems indefinitely near to equality with the
same ordinate limited by the tangent.

If the axis were tangent to the curve, the limit-
ing ratio of area A b d : area A c e would be = L. R.
A d : A e.

If the curve be concave towards the axis, the
tangent lies on a different side of the curve; and the
areas A b d, A c e, by vanishing of the angle between
the chord and tangent, expand into (instead of
shrinking into) coincidence with the rectilinear △*,
A f d, A g e.

LEMMA X.

*The spaces, described from rest by a body acted on by any finite
force, are in the beginning of the motion as the squares of the
times, in which they are described.*

Def. A finite accelerating or retarding force is
such, that the ratio of the time to the velocity
generated or destroyed in that time is finite.

Let the straight line A K represent the time of the body's motion from rest, and K k, drawn at right angles to A K, the last acquired velocity; suppose the time divided into equal intervals A B, B C, C D &c· and let B b, C c, D d &c. drawn at right angles to A K, represent the velocities

Fig. 14.
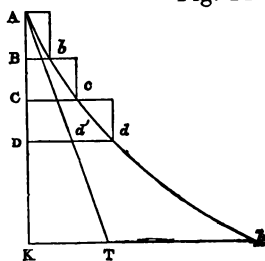
acquired in the times A B, A C, A D &c.; let A $b c d k$ be·the curve passing through the extremities of all the ordinates thus drawn; and complete the parallelograms A b, B c, C d &c.

If now the force be supposed to act by impulses, which would cause the body to move uniformly during the times A B, B C, C D, &c., with the velocities B b, C c, D d, &c., respectively, the spaces described in the 1st, 2d, 3d, &c. intervals will be represented by the parallelograms A b, B c, C d, &c. On this supposition, therefore, the space described in time A D : space in time A K $=$ sum of the parallelograms in the former case : sum in the latter; and this being true always, will be true when the intervals are diminished and their number increased indefinitely, in which case the force, which was supposed to act by impulses, approximates to a continued force, and the sums of the parallelograms to the areas A D d, A K k.

Hence

space in time A D : space in time A K $=$ area A D d : area A K k.

Let the tangent at A cut K k in T; now, the force being finite, the ratio A K : K k is always finite; ∴ A K : K T, which equals L. R. A K : K k is a finite ratio, and therefore,

$$\tan \text{KAT} \left(= \frac{\text{KT}}{\text{AK}} \right) \text{ is finite,}$$

or K A makes a finite angle with the curve at A ;

Hence by Lemma IX,

L. R. area A D d : area A K k = L. R. A D^2 : A K^2,

and therefore in the beginning of the motion, space $\propto$ (time)^2.

By impulse we are to understand a force of the most simple kind, where a single blow or impulse is given to a body, and not repeated. The body by its inertia retains the velocity thus communi-cated, and moves, accordingly, with an uniform velocity; and, supposing absence of all disturbing forces, the space, passed over by this body, would depend on the velocity and the time of its motion, conjointly. Wherefore, as the areas of rectangles are to one another in a ratio compounded of the ratios of their sides, take the two rectangles A b, B c in Fig. 14, for Lemma X. Let A B : B C :: T : t, and let B b : C c as v : V, then area of rectangle A b : area of rectangle B c as v T : V t, as space passed over in time A B with velocity B b uniform through that time, to space passed over in time B C with velocity C c uniform through that time. In the demonstration of the Lemma, the

whole time of the body's motion from rest (repre-
sented by line A K) is divided into equal intervals;
therefore A K is supposed to be divided into equal
successive portions, A B, B C, C D, &c. &c. (or
T = t.) The constant action of the force gives
continued additions of velocity to the body, which
by its inertia retains all these additions; and the
ordinates, drawn from the points B, C, D, &c. re-
present the velocities, which the body has acquired
by the constant action of the force, at the end of
the first interval, the second interval, the third
interval, &c. We must now introduce an incorrect
supposition, that the force, instead of acting inces-
santly, acts by single impulses, a fresh impulse
being given at the commencement of each equal
successive interval. Thus at the commencement
of the first interval A B, an impulse represented by
the ordinate B *b* is given, and, consequently, the
space passed over in the first interval is represented
by the rectangle A *b*. At the commencement of
the second interval B C, a fresh impulse is given,
so that the body now moves with increased
velocity, and this increased velocity is represented
by the ordinate C *c*, and, consequently, the space
passed over in the second interval is represented
by the rectangle B *c*. At the commencement of
the third interval, a fresh impulse is given, so that
the body now moves with increased velocity, re-
presented by the ordinate D *d*, and the space
passed over in the third interval is represented by
the rectangle C *d*; and so on. We have two

errors in our supposition; first, that the force, instead of acting incessantly, acts by single impulses at equal intervals; secondly, that the increment of the velocity, which the body acquires by continued action of force through an interval, is instantaneously communicated to the body at the commencement of the interval. Both these errors are sufficiently corrected by indefinitely increasing the number, and thereby lessening the duration of these equal intervals. The continued impulses do thus indefinitely approximate to action of constant force; and the difference between the sum of the rectangles, and the curvilinear area, traced out by the motion of the ordinate along the axis, does thus indefinitely vanish, so that the portions of this curvilinear area may be taken as proportional measurements of the spaces passed over under the supposed circumstances. If the increments of velocity generated in equal intervals are equal, the constant force is then uniform, $V \propto T$, the ordinates are proportional to their abscissas, the line passing through their vertices is a straight line, and the spaces passed over are then always as the squares of the times. But the constant force may not be uniform, and then the finite portions of the curvilinear area, representing certain spaces, are evidently not in duplicate ratio of the corresponding abscissas of the axis A D and A K. By Lemma IX, that will be the ratio ipso motûs initio, as the action of a *finite* force will necessarily make the axis, representing the time, to meet at a

finite angle the line passing through the vertices of the ordinates representing the velocities. If the axis were a tangent to the curve, the ordinate from a point in the axis (very close to the point of contact) to the curve, would be a line less than any assignable right line, and therefore infinitely small, compared with the corresponding abscissa of the axis, which, though very small, might still be a finite line. But the definition of a finite force gives velocity, generated in any time, always in finite ratio to that time. Therefore, as long as the abscissa of the axis measuring the time be a finite line, so long must the ordinate measuring the velocity be a finite line: consequently, the axis must meet the curve at a finite angle.

To get L. R. A K : K k, suppose both diminished without limit, (i. e. trace both the abscissa and the ordinate back close up to A, there the ratio of the ordinate belonging to the curve : the same ordinate, limited by the tangent, is indefinitely near the ratio of equality. But the ratio of the abscissa to the ordinate, limited by the tangent, is a given ratio, A D : D d', or A K : K T.

Cor. 1. Force is measured by the velocity generated in any time, divided by the time, the force being supposed to remain constant for that time. Hence if D d' be the velocity generated by the force at A, continued constant, in time A D,

$$\text{F at A} = \frac{\text{D}\,d'}{\text{A D}},$$

F

and this being always true, will be true when A D
is diminished indefinitely,

$$\therefore \; F = \text{limit } \frac{D\,d'}{A\,D} = \text{limit } \frac{D\,d}{A\,D} = \frac{K\,T}{A\,K}$$

$$= \frac{K\,T \, . \, A\,K}{A\,K^2} = \frac{2 \text{ triangle } A\,K\,T}{A\,K^2} = 2 \text{ limit } \frac{\text{area } A\,K\,k}{A\,K^2}$$

$$= 2 \text{ limit } \frac{\text{space}}{(\text{time})^2} \; .$$

Ipso motûs initio, the spaces are in the same
ratio, whether constant force be uniform or
variable; i. e. take the velocity that would be
generated in any given time by the force acting
uniformly, and take the velocity that would be
generated by the same force in the same given
time, acting variably, and if the given time be
very small, the difference between these velocities
will be indefinitely small, compared with the
velocities themselves. If A K were tangent to the
curve, K k would vanish before A K. Therefore if
$\frac{K\,k}{A\,K}$ in limit $= o$, the axis is the tangent of a curve.

We cannot see into the abstract nature of
forces, and therefore can only measure or com-
pare them by their effects. Suppose F and f to
be constant uniform forces, and that we are
looking only to their accelerating power. The
velocities generated in a given time (V and v)
would give the ratio of their accelerating power,
or $F : f :: V : v$. Again, as uniform force gives
equal increments of velocity in equal times,

$V \propto T$, and in action of the same uniform force, $\frac{V}{T}$ is invariable; consequently, whether the velocity be generated in 1′ or 5′, or any other number of seconds, if V be that velocity, and T be that number of seconds, $\frac{V}{T}$ would be the measure of F. If the given time be supposed 1″, the expression for the force is most simple, being the velocity generated in 1″. Thus g ($= 32$ feet, 2 inches) represents the accelerating power exerted by the earth's attraction on bodies on or near its surface.

Our main object in the next Section is, to bring into comparison the varying actions of the centripetal force; i. e. to ascertain what law regulates the intensity of a great central attraction. Newton proves in detail by certain proportions and references, that $F \propto L.\ R.\ \frac{sagit}{T^2}$; and then adds, the same thing is easily proved by Lemma X. Turn to Figure 11. and suppose the sagit $=$ D B and T′ $=$ time of the body describing the arc A B. Now if the centripetal force, emanating from a point in the direction A F, had been suspended, when the revolving body reached A, the body would by its inertia have moved on in A D, a tangent to the previous curve at A, and would in T′ have reached D. But the action of F (centripetal force) at A in the direction A F causes it to describe the arc A B in T′. Therefore by the resolution of motion, the subtense D B, parallel to A F, is the space through which F has drawn the body in T′. Consequently

by Lemma X. this space in its limit $\propto$ limit of T^2, even if F be variable. If F and f express different intensities of the central attractive force at different distances from the same centrum virium, $F : f :: \frac{V}{T} : \frac{v}{t}$, therefore $:: \frac{VT}{T^2} : \frac{vt}{t^2}$ supposing F and f to be uniform constant forces, and in that case $VT = 2$ space passed over from rest under the action of F in time T, so as to have V for the last acquired velocity: this also being the case with $\frac{v}{t}$, $F : f :: \frac{2S}{T^2} : \frac{2s}{t^2}$, S and s being spaces passed over in T' and t'. If F and f are variable constant forces, the results are the same, ipso motûs initio, as if the forces were uniform, i. e. the same proportions and equations hold good in the limits of T, V, and S. Thus Prop. VI, $F : f :: L.R. \frac{S}{T^2} : \frac{s}{t^2}$, and taking $F = \frac{V}{T}$, $F = 2$ limit $\left(\frac{\text{space}}{\text{time}^2} \right)$.

Cor. 2. The effect produced by F upon the body is independent of any motion which it may have, when F begins to act upon it. Hence generally if S be the space, through which a force F, acting on a body moving in any orbit, draws the body in T' from the place it would have occupied if the extraneous force had not acted, $F = 2$ limit $\frac{S}{T^2}$.

On the Curvature of Curve Lines.

Supposing, as before, that a curve is a line traced out by a moving point, which is continually

changing the direction of its motion, we see that the more slowly the moving point deviates from its previous direction, the less is the deviation from a rectilinear direction, (i. e. the less is the curvature,) and consequently the less is the angle between the previous direction and the direction immediately subsequent; i. e. the less is the angle of contact, and this angle is taken as the measure of curvature. Any two such angles are less than any assignable rectilinear angles; but we measure them by regarding them as limits of certain rectilinear angles, taking the equations for such angles, and ascertaining the L. R. of such equations. Thus in Fig. 15. Prop. I. the angles of contact at P and p are the limits of the angles between chord and tangent, and these angles in their limit $\propto$ their sines.

Prop. I. If in P R, pr tangents at the points P, p in the curves P Q, pq, P R be taken equal to pr, and the subtenses Q R, qr be drawn equally inclined to them, then when Q R, qr move parallel to themselves to P, p,

Fig. 15.
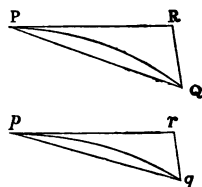
$$\frac{\text{curvature of PQ at P}}{\text{curvature of } pq \text{ at } p} = \text{limit } \frac{QR}{qr}.$$

Draw the chords P Q, $p\,q$,

then $\dfrac{\text{curvature of P Q at P}}{\text{curvature of } p\,q \text{ at } p} = \dfrac{\text{angle of contact at P}}{\text{angle of contact at } p}$

$$= \text{limit } \frac{\text{angle Q P R}}{\text{angle } p\,q\,r}$$

the $\angle^{\cdot}$ between chord and tangent.

$$= \text{limit}\ \frac{\sin \text{Q\,P\,R}}{\sin \ q\,p\,r}$$

(since angles $\propto$ their sines when very small.)

(By Lemma VII. $\text{PQ} = \text{RP}$) $= \text{limit}\ \dfrac{\dfrac{\text{Q\,R}}{\text{R\,P}}\ \sin \text{R}}{\dfrac{q\,r}{r\,p}\ \sin r}$

$$= \text{limit}\ \frac{\text{Q\,R}}{q\,r}\ .$$

Prop. II. The curvatures in different circles vary inversely as the diameters.

Let PQV, *pqv* be two circles, draw the diameters PV, *pv*, and the tangents PR, *pr*. Take P R $= p\,r$, and draw the sub-tenses Q R, *q r* parallel to the diameters, and Q N, *qn* parallel to the tangents;
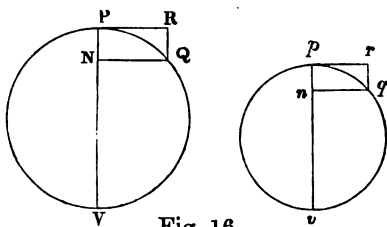
Fig. 16.

$$\text{then }\ \frac{\text{Q\,R}}{q\,r} = \frac{\text{PN}}{p\,n} = \frac{\text{QN}^2}{\text{NV}} \div \frac{q\,n^2}{n\,v} = \frac{n\,v}{\text{NV}}\ ,$$

$$\therefore\ \frac{\text{curvature at P}}{\text{curvature at } p} = \text{limit}\ \frac{\text{Q\,R}}{q\,r}$$

$$= \text{limit}\ \frac{n\,v}{\text{NV}}$$

$$= \frac{p\,v}{\text{PV}}$$

or the curvature $\propto \dfrac{1}{\text{diameter}}\ .$

Cor. Hence in the same circle the curvature is the same at every point.

From this property of the circle, and also because by varying the diameter it may be made to have any curvature we please, the circle is made use of to measure the curvature at any proposed points of other curves.

Def. The *circle of curvature* at any point of a curve is that circle, which has the same tangent and curvature as the curve has at that point.

Hence if $Q q R$ be a common subtense to the curve PQ and the circle Pq, and limit $\frac{QR}{qR} = 1$, Pq will be the circle of curvature at P.

Fig. 17.

The radius, diameter, and chord of the circle of curvature are generally called the radius, diameter, and chord of curvature.

Prop. III. If PqV be the circle of curvature at any point P, and PV a chord drawn in any given direction, then

Fig. 18.

$$PV = \text{limit} \frac{(\text{arc})^2}{\text{subtense parallel to the chord}}.$$
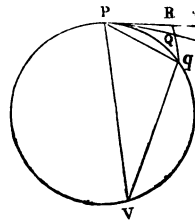
Take P Q a small arc of the curve, through Q draw the subtense RQq parallel to PV, and join Pq, qV; then since the triangles PRq, PqV are evidently similar,

$$PV = \frac{Pq^2}{qR}$$

Now this being true, whatever be the magnitude of PQ, will be true, when RQr moves parallel to itself up to P, in which case $Pq = PQ$ ultimately, and $qR = QR$ ultimately,

$$\therefore PV = \text{limit } \frac{Pq^2}{qR}$$

$$= \text{limit } \frac{(\text{arc } PQ)^2}{QR} .$$

Cor. Hence the diameter of curvature

$$= \text{limit } \frac{(\text{arc})^2}{\text{subtense perpendicular to the tangent}} .$$

Prop. IV. If in the curve PQ, PG, and QG, drawn perpendicular to the tangent PR and the chord PQ respectively, intersect in G, then when Q moves up to P, the limit of PG is the diameter of curvature at P.
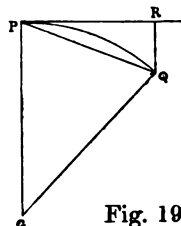
Fig. 19.

Draw the perpendicular subtense QR, then by similar triangles PQR, PGQ

$$PG = \frac{PQ^2}{QR} ;$$

$$\therefore \text{limit } PG = \text{limit } \frac{PQ^2}{QR}$$

$$= \text{limit } \frac{(\text{arc } PQ)^2}{QR}$$

$$= \text{diameter of curvature at P.}$$

Def. The curvature of a curve at any point is said to be finite, when the diameter of curvature at that point is finite.

LEMMA XI.

In curves of finite curvature, the limiting ratio of the subtenses equals that of the squares of the conterminous arcs.

Let A bB be the curve having a finite curvature at A;

First, Let the subtenses bd, BD, be perpendicular to the tangent at A. Draw bg, B G at right angles to the chords Ab, A B, and let them meet AgG, which is drawn at right angles to the tangent A D, in the points g and G.
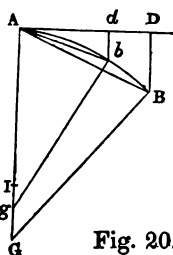
Fig. 20.

Then as b and B move up to A, g and G move up to I, the extremity of the diameter of curvature of A, as their limit. (Prop. IV.)

Now by similar triangles,

$$ BD = \frac{AB^2}{AG}, \quad bd = \frac{Ab^2}{Ag}, $$

$$ \therefore BD : bd = \frac{AB^2}{AG} : \frac{Ab^2}{Ag}, $$

$$ \therefore \text{L. R. } BD : bd = \text{L. R. } \frac{AB^2}{AG} : \frac{Ab^2}{Ag} $$

$$ = \text{L. R. } AB^2 : Ab^2, $$

(since A G, Ag, are ultimately equal to A I)

$$ = \text{L.R. (arc AB)}^2 : \text{(arc A}b)^2. $$

Because G I may be assumed less than any assignable length, the ratio of A G to Ag may differ from the ratio of equality, less than by any assignable difference.

Secondly, Let the subtenses be inclined at any
equal angles to the tangent.
Draw B E, *b e,* perpendicular
to the tangent: then by simi-
lar triangles,
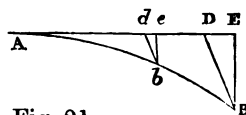
Fig. 21.

$$BD : BE = bd : be,$$

alternando $BD : bd = BE : be$;

$$\therefore \text{L. R. } BD : bd = \text{L. R. } BE : be$$
$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$

Thirdly, Let the subtenses, inclined at unequal
angles to the tangent, converge to a point, and
revolve round that point fixed, or approach to A
according to any other given law.

Let O be the point in which D B, *d b,* meet
when produced.   The angles
D and *d,* being determined by
the same law, will always con-
verge to equality ; for the arcs
A B, A *b,* are diminished with-
out limit, and therefore their
difference, the arc B *b,* will be

Fig. 22.

diminished without limit ; thus the angle B O *b*
must be diminished without limit ; but this angle
$= \angle AdO - \angle ADO$.   Thus these angles, AdO
and A D O, are ultimately equal, and this third case
comes under the second case.
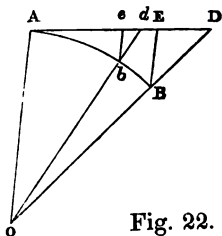
Cor. 1.   Hence by Lemma VII. the limiting
ratio of the subtenses will equal that of the squares
of the arcs, chords, and tangents.

Theorem. *The subtense of an arc is ultimately equal to four times the parallel sagitta.*

Def. The sagitta of an arc is a line drawn at a finite angle to the chord from its middle point to meet the arc.

Let BD be a subtense of the arc AB, EC the sagitta parallel to it, bisect-
ing the chord in E, and pro-
duced to meet the tangent
in F.
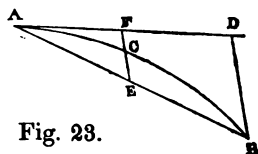
Fig. 23.

Then by similar triangles,

$$AF = \tfrac{1}{2} AD, \text{ and } EF = \tfrac{1}{2} BD.$$

Also by the Lemma,

$$\text{L. R. } CF : BD = \text{L. R. } AF^2 : AD^2$$
$$= 1 : 4$$
$$\therefore \text{L. R. } CE : BD = 1 : 4.$$

Cor. 2. The limiting ratio of the sagittæ, which bisect the chords and con-
verge to a given point,
equals that of the squares
of the arcs, chords, and
tangents.
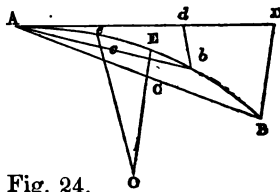
Fig. 24.

Let EC, ec, be the sagittæ of the arcs AEB, Aeb, bisecting the chords AB, Ab in C, c; draw the subtenses BD, bd respectively parallel to them;

Then $\text{L. R. } EC : BD = 1 : 4$
$$= \text{L. R. } ec : bd;$$

$\therefore$ L.R. EC:ec = L.R. BD : bd;

$\quad\quad\quad$ = L.R. (arc AB)^2 : (arc Ab)^2

$\quad\quad\quad$ = L.R. (chord AB)^2 : (chord Ab)^2

$\quad\quad\quad$ = L.R. (tangent AD)^2:(tangent Ad)^2.

Cor. 3. Hence if a body describe the arcs AB, Ab, with any given velocity, the limiting ratio of the sagittæ will be that of the squares of the times, in which they are described.

Cor. 4. If the subtenses DB, db, be perpendicular to the tangent, as in the first case of the Lemma,

$\quad\quad$ $\triangle$ADB : $\triangle$Adb = AD . DB : Ad . db;

$\therefore$ L.R. $\triangle$ADB : $\triangle$Adb $\quad$= L.R. AD·. DB : Ad.db

$\quad\quad\quad\quad\quad\quad\quad$ = L.R. AD^3 : Ad^3

$\quad\quad\quad\quad\quad$ or $\quad$= L.R. DB$^{\frac{3}{2}}$: $db^{\frac{3}{2}}$.

Cor. 5. Since L.R. DB : db = L.R. AD^2 : Ad^2, the limiting form to which every curve of finite curvature approximates is the common parabola.

Hence L.R. area ADB : area Adb

$\quad\quad\quad$ = L.R. $\frac{2}{3}$ AD . DB : $\frac{2}{3}$ Ad . db

$\quad\quad\quad$ = L.R. AD^3 : Ad^3

$\quad\quad$ or $\quad$= L.R. DB$^{\frac{3}{2}}$: $db^{\frac{3}{2}}$.

Conic Sections give us the entire parabolic area = $\frac{2}{3}$ circumscribed parallelogram. Therefore in Fig. of first case the curvilinear area ADB = (ultimately) $\frac{2}{3}$ AD . DB.

SCHOLIUM TO LEMMA XI.

It was proved in the Lemma, that if the curvature be finite, the subtense varies ultimately as the square of the conterminous arc ; conversely,

If the subtense vary ultimately as the square of the arc, the curvature is finite, and if it vary according to any other power of the arc, the curvature is infinitely great or infinitely small.

Let P Q and P q be arcs of a curve and circle, having a common tangent P R, and let R Q q be a common subtense.
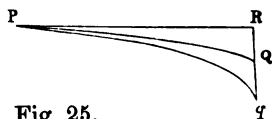
Fig. 25.

Since in the circle q R $\propto$ ult. P R^2, let q R $= a$. P R^2 ultimately,

and suppose that Q R $\propto$ ult. P R^n and $= b$. P R^n ultimately

$$\therefore \frac{\text{curvature of P Q}}{\text{curvature of P } q} = \text{limit } \frac{\text{Q R}}{q \text{ R}} = \frac{b}{a} \cdot \text{limit P R}^{n-2}.$$

If $n = 2$, the curvature of the curve P Q bears a finite ratio to that of the circle, and is therefore finite. If n be greater than 2, limit P R$^{n-2} = 0$, and therefore the curvature of P Q is infinitely small compared with that of P q, and the curve will lie between P q and the tangent. If n be less than 2, limit P R$^{n-2} = \infty$, and therefore the curvature of P Q is infinitely great, and the curve will lie below P q.

The cases here given suppose $q \, \mathrm{R} : \mathrm{P} \, \mathrm{R} :: \mathrm{P} \, \mathrm{R} :$ $\frac{1}{a}$, and $\mathrm{Q} \, \mathrm{R} : \mathrm{P} \, \mathrm{R}^{n-1} : \mathrm{P} \, \mathrm{R} : \frac{1}{b}. \;\therefore\; q \, \mathrm{R} = \mathrm{P} \, \mathrm{R}^2 \, a,$ and $\mathrm{Q} \, \mathrm{R} = \mathrm{P} \, \mathrm{R}^n \, b. \;\therefore\; \mathrm{Q} \, \mathrm{R} : q \, \mathrm{R} :: \mathrm{P} \, \mathrm{R}^n \, b : \mathrm{P} \, \mathrm{R}^2 \, a$ $:: \mathrm{P} \, \mathrm{R}^{n-2} \, b : a, \; :: b : a,$ if $n = 2,$ as then $\mathrm{P} \, \mathrm{R}^{n-2}$ $= 1.$ But if n be greater than 2, $\mathrm{P} \, \mathrm{R}^{n-2}$ is positive, and its limit is $o,$ as Q and q move up towards coincidence with P; therefore, in this case, L. R. $\mathrm{Q} \, \mathrm{R} : q \, \mathrm{R} :: o \, . \, b : a.$ If n be less than 2, then $\mathrm{P} \, \mathrm{R}^{n-2}$ is negative, and L. R. $\mathrm{Q} \, \mathrm{R} : q \, \mathrm{R} :: \frac{b}{o} : a.$

Cor. Since an infinite number of values may be given to $n,$ to each of which there will be a corresponding curve, an infinite number of curves may be described between $\mathrm{P} \, q$ and the tangent, corresponding to values of n greater than 2, and an infinite number below $\mathrm{P} \, q,$ corresponding to values of n less than 2.

Suppose n to go on increasing, and to become $n + a, n + 2 \, a,$ &c. each corresponding curve will have a curvature indefinitely less than the curvature of the preceding curve. And if, on the other hand, n go on decreasing in value, each corresponding curve will have a curvature indefinitely greater than the curvature of the preceding curve.

The greater the value of $n,$ expressing that power of the tangent $\mathrm{P} \, \mathrm{R}$ which ultimately gives the ratio of the decreasing subtenses to the same angle of contact, the more rapidly are those subtenses decreasing, and, consequently, the more

rapid is the approximation of the curve to co-incidence with the tangent; and therefore, tracing it the other way, in its generation from the point of contact, the slower must have been its deviation from the direction of the tangent. Angles of contact, though incapable of comparative measurement with rectilinear angles, have still a definite ratio among themselves, which is expressed by the ultimate ratio of their subtenses. This result is obtained by considering these angles as the limits of certain rectilinear angles, and this gives the subtenses ultimately as the sines of such angles.

In these angles of contact, and on other occasions in mathematical reasonings, we meet with infinitesimal quantities, between which and finite quantities we cannot establish a precise proportion. But where we can consider these infinitesimals as limits of certain finite quantities, we can ascertain the ratio existing between these infinitesimals themselves. As we meet with infinitesimals in our abstract reasonings, analogy would lead us to expect infinitesimals of various descriptions in the material world, whose existence and agency may seem beyond our exact cognizance. The atoms of matter, taken individually, are infinitesimals. Again, in bodies called solid, we suppose the constituent atoms in immediate contact, whereas there are intervals of space between them. These intervals are infinitesimal portions of space, which we cannot bring into exact comparison with our measurements of distance ; but these intervals are in

certain ratios to one another; and as the power of attraction acts inversely in duplicate ratio of the distance, the particles of a body may be cohering, with more or less tenacity, (i. e. the body may seem more or less hard,) in the duplicate of the ratio, which is really existing between these infinitesimal distances of the atomic particles.

SECTION II.

PROP. I. *If a body move in any orbit about a fixed centre of force,
the areas, described by lines drawn from the centre to the body,
lie in one plane, and are proportional to the times of describing
them.*

LET S be the centre of force ; and suppose a
body unattracted by the
the force in S to describe
the straight line A B with
a uniform velocity in a
given time (T).   Then if
suffered to proceed, it
would move on uniformly
in the direction of A B
produced,   and   describe

Fig. 26.

B c = A B in the next interval (T); but at B
suppose an instantaneous impulse communicated
to it in the direction B S, which causes it to move in

G
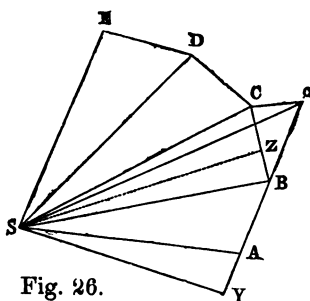
the direction B C ; draw c C parallel to B S, then by
the principles of Mechanics, the body at the end of
the second interval will be found at C.   Join S A,
S B, S c, S C.   Since c C is parallel to B S, the
triangle S B C = S B c = S A B, since B c = A B;
and these triangles are in the same plane, as no
force has acted to draw the body out of the plane
S A B.   Similarly, if impulses be communicated at
the end of every interval of T', in directions tending
always to S, causing the body to describe C D,
D E, &c. in the third, fourth, &c. intervals, the
triangles S A B, S B C, S C D, &c. will be all equal,
and will lie in the same plane ; and their bases
A B, B C, C D, &c. are described in equal times,
therefore the area of any number of these triangles,
or the polygon S A B C D E varies as the time of
describing it.   Now let the number of intervals be
increased, and the magnitude of each diminished
indefinitely, then the polygon approximates to a
curvilinear area, and the sum of the impulses to a
continued force always tending to S, as their limits ;
and what was proved of those quantities is true of
their limits, and therefore the curvilinear area
described in any time is proportional to the time.

Obs. The area, described by the line joining S
and the body, is frequently called the area de-
scribed by the body round S.

The space B C, which is supposed to be passed
over in the second equal interval T, is given in

the direction intermediate between the previous direction B.c of body, and the direction B S of centripetal impulse; and the length of B C is determined by the point where the line from c, parallel to the centripetal force, intersects it.

We assume an error in our hypothesis, that the attraction from S acts by impulses at the end of each equal interval T, and not incessantly. This erroneous supposition leads easily to the conclusion, that the body moves in equal intervals along the sides A B, B C, C D, &c. of the polygon. Then indefinitely increasing the number of the intervals, and shortening the duration of each, we bring the action of the impulses indefinitely near to the action of constant force, and the perimeter (made up of the sides A B, B C, C D, &c.) indefinitely near to a curve. If a body move in a curve, this of itself is proof of the action of a constant force.

This proposition, and its converse the next, give a most important step in Physical Astronomy. Kepler, by the aid of Tycho Brahe's observations, had ascertained the fact, that the law of the motion of each planet round the sun is, that it describes equal areas about the sun in equal times. And Newton here shews that this law is a necessary consequence, by reference to the first and second laws of motion, of a constant attracting force emanating from a central point, and acting on a body to which a projectile impulse had originally been given. The way was thus opened for sub-

sequent observations and calculations, which have
been successively and successfully carried on, in
proving that the various· and intricate motions
within our planetary sphere depend only on a
primary motion of projection, and the simple law
of gravity.

We have now two points to ascertain ;—what
general law for the variation of the velocity, and
what law for the variation of the centripetal force,
will arise from the fact of equal areas in equal
times in the same orbit.

Cor. 1. The velocities in the same orbit at the points
A, B, C, &c. will be reciprocally, as the perpendicu-
lars, let fall from the centre of force, on the tangents
of the curve at the points A, B, C, &c. S Y, drawn
from S, (fig. 26.) perpendicular on A B, is the
perpendicular height of the triangle A S B ∴ A B .
S Y = 2 △ A S B ; for the same reason (if S Z were
drawn perpendicular on B C), C B . S Z = 2 △S C B.
But the △'s A S B, B S C, are equal ; therefore the
rectangles A B . S Y and B C . S Z are equal ; and
∴ A B : B C :: S Z : S Y.   Now velocity at A is
supposed to be continued unchanged till the body
reaches B ; it is then perhaps changed, but changed
by a single impulse given at B ; therefore this
changed velocity will be continued uniform, till the
body reaches C.   Thus A B : B C (being spaces
passed over in equal times with uniform velocities)
is as velocity at A to velocity B :: S Z : S Y.   All
this will hold true in the limits ; but ultimately

A B will coincide with the tangent at A, and B C with the tangent at B. Thus in the same orbit the velocity at any point $A \propto \dfrac{1}{\text{Perpendic. S Y at A}}$. One orbit will not probably have the area described dato tempore = area dato tempore in another orbit. But if these areas are evanescent, the little arcs, which are their bases, may be taken as measures of the velocities, with which these areas are described. And the arc . × perpendicular from the centre of force upon it = 2 area. Thus the arc $= \dfrac{2 \text{ area}}{\text{Perpendic.}}$ or velocity $\propto \dfrac{\text{area dato tempore}}{\text{S Y}}$.

This expression is general, giving the ratio of velocities at points in different orbits, whether round the same or different centres of force.

Cor. 2. If the chords A B, B C, of two arcs, successively described in equal times by the same body in spaces void of resistance, are completed into a parallelogram A B C V, and the diagonal B V of this parallelogram,
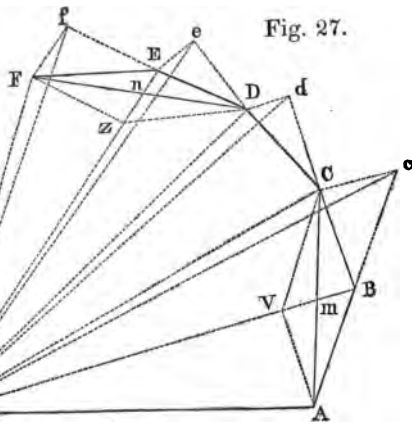
Fig. 27.

in the position which it ultimately acquires, when those arcs are diminished indefinitely, is produced both ways, it will pass through the centre of force.

C V being drawn equal and parallel to B A, is so to B c. Wherefore C V and B c being equal and parallel, will be terminated by lines equal and parallel, ∴ B V must be equal and parallel to c C, which is drawn parallel to the direction of the attractive impulse, supposed to be impressed from centre S on the body at B, ∴ B V produced passes through S.

Cor. 3. If the chords A B, B C, and D E, E F, of arcs, described in equal times in spaces void of resistance, are completed into the parallelograms A B C V, D E F Z; the forces in B and E are to each other in the ultimate ratio of the diagonals B V and E Z, when those arcs are diminished indefinitely. For the motions B C, and E F of the body are compounded of the motions B c, B V, and E f, E Z: but B V and E Z, equal to C c and F f, in the demonstration of this proposition, were generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

Cor. 4. The forces, by which bodies in spaces void of resistance are drawn back from their rectilinear motions, and turned into curvilinear orbits, are to each other, as those sagittæ of arcs

described in equal times, which converge to the centre of force, and bisect the chords, when those arcs are diminished indefinitely. For such sagittæ are half the diagonals mentioned in Cor. 3.

The diagonals C A and F D being drawn, they bisect the other diagonals B V and E Z ; and the upper halves of these two last diagonals will be the sagittæ of the arc made up of A B, B C, and of the arc made up of D E, E F, and will bisect the chords A C and D F of those arcs.

Cor. 5. And, therefore, those forces are to the force of gravity, as the said sagittæ to the sagittæ perpendicular to the horizon of the parabolic arcs, which projectiles describe in the same time.

Fig. 28.
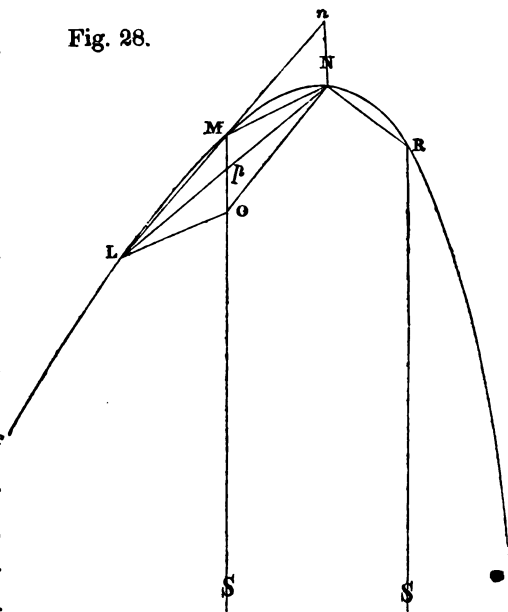
Obs. 1. In the case of these projectiles, the constant force, which keeps drawing the projectile out

of its rectilinear direction, is gravity, i. e. the earth's attraction ; and this acts in directions perpendicular to the horizon.   Consequently, if two successive arcs, described in very small equal times, of the parabolic curve described by the projectile, as in Fig. 28, had their chords drawn and completed into a parallelogram, the diagonal of this parallelogram would be perpendicular to the horizon, and would represent the space through which gravity would draw the body downwards in the very small supposed time, supposing gravity acted by impulses at the ends of these small supposed intervals ; and the halves of these diagonals would be the sagittæ mentioned in the Cor.

Obs. 2. If a constant force act uniformly on a body for a given time, say 1′, it draws the body through a space, equal to half the velocity generated in the body by the uniform action of the force continued through that given time.   Thus gravity, acting freely on a body for 1′, generates in the body a velocity which would carry the body in 1′ over thirty-two feet, two inches, and it will have drawn the body through sixteen feet, one inch.   Wherefore, if that velocity were all at once, uno magno impulsu, given to the body, and the body were then left to move freely, without further acceleration or disturbance, it would in 1′ pass down thirty-two feet two inches, and this space would be represented by the entire diagonal B V. Whereas if the constant force were only so acting,

as by continued action for 1′ to generate this velocity, the space, through which the force draws the body in the 1′, would be represented by the sagit, or half the diagonal.

Obs. 3. The expressions for centripetal forces in Cor. 3 and 4 are general, and may be applied to obtain the ratio of centripetal forces at different points in the same or different curves, as they have reference only to the spaces actually passed over, or the velocities actually generated, by the forces supposed to act uniformly for a given time. The general principle here is, to measure the cause by its real legitimate effect in a given time,—the cause being supposed to act uniformly for that time.

PROP. II. *If a body, moving in a curve, describe in one plane areas proportional to the times by lines drawn from the body to any point, the body is acted on by centripetal forces all tending to that point.* (Vide Fig. Prop. I.)

Let S be the point, about which areas proportional to the times are described; and suppose as in Prop. I. that a body, unattracted by the force in S, describes the straight line A B in a given time T.

In A B produced take B *c* = A B; then if suffered to proceed, the body would be at *c* at the end of the second interval of T′. But at B suppose an impulse communicated, which causes it to

describe B C in the second interval, such that the triangle S B C = S A B.   Join c C, S c.

Then the triangle S B C = S A B = S B c, therefore c C is parallel to B S, and therefore by the principles of Mechanics the impulse communicated at B tends to S.   Similarly if D, E, &c. be the places of the body at the ends of the third, fourth, &c. intervals of T', so that the triangles SAB, SBC, SCD, &c. are all equal, all the impulses communicated may be shewn to tend to S.

Now suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the polygon is the curvilinear area, and that of the sum of the impulses a continued force tending to S; and the above reasoning still holds in the limit, therefore the body is acted on by a continued force tending to S.

PROP. III. *Cor.* Draw C V parallel to A B meeting S B in V and join A V.   Then C V = B c = A B, ∴ A V is equal and parallel to C B, or A B C V is a parallelogram.   Draw the diagonal C A bisecting B V in *m*.

Now suppose S′ A′ B′ C′ D′ to be another orbit, in which the chords A′ B′, B′ C′ are described in the same time as either of the chords A B or BC; and let the same construction be made as in the former orbit, then

Fig. 29.
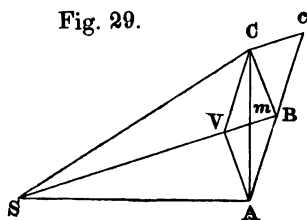
impulse at B : impulse at B' $= c$ C : c' C' $=$ B m : B'm', and therefore force at B : force at B' $=$ L. R. B m : B'm'; or the centripetal forces in different orbits are in the limiting ratio of the sagittæ of arcs described in equal times, which ultimately pass through the centres of force.

This is termed Prop. III. by Evans, as he omits Cor. 3 and 4 of Prop. I. of which Cor. this Prop. III. is the substance.

PROP. IV. *The centripetal forces, by which bodies describe different circles with uniform velocities, tend to the centres of the circles, and are as the squares of the arcs, described in the same time, divided by the radii.*

Since in each circle the motion is uniform, the arcs described are proportional to the times. But the sectors, i. e. the areas described, are as the arcs on

Fig. 30.

which they stand; and are therefore proportional to the times. Hence (Prop. II.) the forces tend to the centres of the circles.
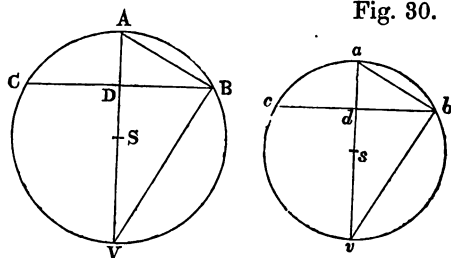
Again, let C A B, $c\,a\,b$, be arcs described in the same time in the circles, whose centres are S, s, and let A, a, be their middle points; join A B, $a\,b$,

and draw the diameters ASV, asv cutting the chords CB, cb in D, d; then (Prop. II. Cor.)

Force at A : force at $a = $ L. R. $AD : ad$

$$= \text{L. R. } \frac{(\text{chord } AB)^2}{AV} : \frac{(\text{chord } ab)^2}{av}$$

$$= \text{L. R. } \frac{(\text{arc } AB)^2}{AS} : \frac{(\text{arc } ab)^2}{as}.$$

Take AE, ae any other arcs described in equal times;

then $AE : ae = AB : ab$,

and this being true whatever be the magnitudes of AB, ab will be true when they are diminished indefinitely,

$$\therefore AE : ae = \text{L. R. } AB : ab,$$

and therefore force at A : force at $a = \dfrac{AE^2}{AS} : \dfrac{ae^2}{as}$.

As diameter AV bisects the arc CAB, it must bisect the chord of that arc, and Cor. 2. 3. 4. of Prop. I. prove that it must pass through the centre of force, and thus in conjunction with Prop. II. prove that the centre of the circle is the centre of force.

From the manner in which the figures are drawn, AD and ad are sagits of two simultaneous arcs CAB and cab; and $\therefore$ by Cor. 4. of Prop. I. and by Prop. III. are ultimately as the centripetal forces at A and a.

The circumstances supposed in this Proposition, enabled us to express the ratio of the centripetal forces by finite magnitudes.

Prop. IV. *Cor.* 1. Supposing the planets to be revolving in circular orbits with uniform velocities round the sun as the common centre, this Proposition would enable us to ascertain in what ratio the centripetal force, drawing the planets out of a rectilinear direction, was affected by the different distances of the planets from the common centre of force. Take two of these supposed circular orbits, let A E and $a\,e$ be two simultaneous arcs; (Fig. 30.) R and r the two radii of these orbits; and F and f the two centripetal forces. Then $F : f : : \dfrac{AE^2}{R} : \dfrac{ae^2}{r}$. But the two arcs being spaces described in the same time with uniform velocities are as these velocities $\therefore F : f : : \dfrac{V^2}{R} : \dfrac{v^2}{r}$. Wherefore if we knew the velocities and the radii of the two orbits, we should get the ratio of $F : f$.

Cor. 2. The periodical times of the two orbits are easier to be obtained than their velocities. Therefore we now take the value of the velocity in terms of the periodical times. Where two spaces are described with uniform velocities, these velocities are to one another as the spaces directly and the times of description inversely; and in the case before us, the spaces being the circumferences of the circles are as the radii $\therefore V^2 : v^2 : : \dfrac{R^2}{P^2} : \dfrac{r^2}{p^2}$. Substituting this ratio for $V^2 : v^2$, we now have $F : f : : \dfrac{R^2}{P^2 . R} : \dfrac{r^2}{p^2 . r} : : \dfrac{R}{P^2} : \dfrac{r}{p^2}$.

Cor. 3. We must next enquire whether the periodic times have any fixed relation or ratio to the radii of the planetary orbits. If $P = p$, then $F : f : : R : r$, and $V : v : : R : r$. In such a case, the centripetal force could not be the general power of attractive influence exerted by the sun, as such influence is lessened, not increased by the greater distance of the attracted body. ·

Cor. 4. If it had turned out on observation that $P : p : : R^{\frac{1}{2}} : r^{\frac{1}{2}}$, then we should have $V : v : : R^{\frac{1}{2}} : r^{\frac{1}{2}}$ and $F = f$, and this would have been fatal to Newton's theory of gravity.

Cor. 5. If $P : p : : R : r$, this would give $V = v$, and $F : f : : r : R$. And this would give the centripetal force acting more powerfully at the less distance.

Cor. 6. Kepler had ascertained, that the periodical times of the planets were in the sesquiplicate ratio of their mean distances. Now if in our supposed circular orbits, $P^2 : p^2 : : R^3 : r^3$, then $V : v : : \frac{1}{R^{\frac{1}{2}}} : \frac{1}{r^{\frac{1}{2}}}$, and $F : f : : \frac{1}{R^2} : \frac{1}{r^2}$.

Cor. 7. And universally if $P : p : : R^n : r^n$, then $V : v : : \frac{1}{R^{n-1}} : \frac{1}{r^{n-1}}$, and $F : f : : \frac{1}{R^{2n-1}} : \frac{1}{r^{2n-1}}$.

The eighth Corollorary of Newton's text can be better illustrated under Prop. VI. as is done by Evans.

Cor. 9. The arc, which a body, uniformly revolving in a circle with a given centripetal force,

describes in any time, is a mean proportional
between the diameter of the circle and the space,
which the same body, descending by the same
given force, would describe in the same given time.

Let P A be an arc described
in any time, P B the space
fallen through in the same
time by the force at P con-
tinued uniform; take PQ an
evanescent arc, QR the sub-
tense parallel to P S, the
radius from P to S the centre

Fig. 31.
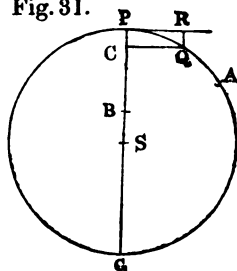
of force and of the circle, and complete the paral-
lelogram. Then the evanescent subtense Q R or
P C is the space fallen through by the centripetal
force in the same time that the arc PQ is described.
For the tangent P R is the rectineal direction, in
which the body by its vis inertiæ would move;
wherefore being found at Q, its motion (the arc
P Q) in time t is by resolution of motion, made up
of P R and R Q, R Q being the direction in which
the constant force from S (the centripetal force) is
during time t pulling the body. The time t must
be indefinitely small to allow of the supposition,
that through its continuance the constant force
from S is pulling the body in a direction parallel to
P C. Let T represent the time of falling through
PB, as t represents the time of falling through PC;
then P C and P B being spaces fallen through by
action of the same constant uniform force are as
$t^2 : T^2$. But the arcs P Q and PA are as the times

of their description; thus $PC : PB :: t^2 : T^2 :: PQ^2$ $: PA^2 :: \frac{PQ^2}{PG} : \frac{PA^2}{PG}$. But at the commencement of this Prop. we saw that $PC = \frac{PQ^2}{PG} \therefore PB = \frac{PA^2}{PG}$, and $PB : PA :: PA : PG$.

SCHOLIUM.

Under his definition of centripetal force, Newton states the supposition of a cannon ball being shot off at a small distance above the surface of the earth, and in the direction parallel to the horizon. The ball describes a slanting curve, and according as a greater velocity is given, it flies further and further before it is drawn down to the ground by the earth's attraction. We can conceive the velocity to be very much increased, and consequently the effort of the ball for persevering in a rectilinear direction to become very powerful, so that it flies very many miles before it is drawn down to the ground. And we may conceive this increase of velocity to be so great, that the ball shall be so very slowly drawn out of its rectilinear direction, as owing to the spherical shape of the earth, never to be drawn nearer to the earth's surface, but to circulate round the earth like a little moon. Again, if the velocity of the ball were still further increased, we see that the earth's attraction would not draw it sufficiently fast downwards to prevent it, owing to the earth's spherical form, from receding from the earth. All this supposes no resisting medium of atmosphere. And

we see there must be a certain balance between velocity of projection and power of centripetal force, to account for a body revolving in a circle round a centre of force. That balance is easily ascertained from the last Corollary and its figure. To move in a given circle round a given centre of force, the body must be projected in a tangential direction, and with such a velocity, as the centripetal force, acting uniformly, would generate in the body, if it moved it from rest through half radius of the circle. Thus suppose P B to be the space that would give the required velocity; (fig. 31.) i. e. that the given centripetal force moving a body from rest through a space $=$ P B, would by this continued action generate in the body the velocity required for its circulation in the given circle. Suppose P A to be the arc of the given circle uniformly described in time, equal to the time of the body's fall through P B; then P A $=$ 2 P B, on the principles of " uniformly accelerated motion." But (Cor. 9. Pr. IV.) P B : P A : : P A : P G, i. e. P B : 2 P B : : 2 P B : P G, i. e. 4 P B $=$ P G or P B $= \dfrac{P G}{4}$. Thus in the case just supposed of the cannon ball gravity, or earth's attraction at its surface, is the centripetal force, the circle $=$ circumference of earth, and the velocity must be such as gravity would generate in a body, if that body could fall from rest uninterruptedly down a perpendicular $= \frac{1}{4}$ of earth's diameter $=$ about 2000 miles. Newton says, " by

H

means of Prop. IV. and its Corollaries, we may dis-
cover the proportion of a centripetal force to any
other known force, such as that of gravity.   For if
a body by means of its gravity revolves in a circle
concentric to the earth, this gravity is its centri-
petal force.   But from the descent of heavy bodies,
the time of one entire revolution, as well as the
arc described in any given time, is given by Cor. 9.
of Prop. IV." In comparing two centripetal forces
by Prop. IV. our two formulæ are $F : f :: \dfrac{V^2}{R} : \dfrac{v^2}{r}$
or as $\dfrac{R}{P^2} : \dfrac{r}{p^2}$.   The orbit of gravity alluded to by
Newton, is the orbit of the cannon ball.   Conse-
quently we must get the velocity and periodic time
belonging to it.   By Cor. 9. the arc, described in
1′, i. e. the velocity, is a mean proportional between
space, through which a heavy body falls in 1′, and
diameter of earth $= \sqrt{}$ of product of $16\frac{1}{12}$ feet into
number of feet in earth's diameter (vel. $= \sqrt{\frac{g}{2} . 2\,r}$)
this gives velocity of near five miles in 1′.

Again, as five miles : number of miles in earth's
circumference : : 1′ : number of seconds in P,
would be a rough way of calculating the periodical
time of the cannon ball $=$ 1h. 24min. 27′.   We
may simplify the expression for the velocity by
using the formula, $S \propto V^2$ and $V^2 = 2\,F\,S$, esta-
blished in uniformly accelerated rectilinear motion.
Forces being represented by the velocity they
generate in 1′, g or gravity $= 32\frac{1}{6}$ feet, and $\dfrac{g}{2}$ or
$16\frac{1}{12}$ feet is the space fallen through to acquire that

velocity $\therefore \frac{g}{2} : \frac{r}{2} :: g^2 : v^2$ the square of the velocity of the cannon ball. $\therefore v^2 = rg$, and $v = \sqrt{gr}$ expressed in feet; this simplifies the expression for the periodic time, as such time

$$= \frac{\text{the whole circumference}}{\text{vel. or (arc in 1')}}$$

putting $\pi = 3 . 14159$, i. e. circumference of circle, whose diameter $= 1$, and circumferences of circles being as their radii, $2 r \pi = $ whole circumference, and $\therefore$ periodic time $= \frac{2 \pi r}{\sqrt{gr}} = \pi . \sqrt{\frac{4r}{g}}$ in seconds, r being expressed in feet.

To compare this centripetal force of gravity at the earth's surface with centripetal force (F) in any other circular orbit, by Prop. IV. of this Section, we must have the radius (R) of that orbit, and its velocity (V) or its periodic time; then $F : g :: \frac{V^2}{R} : \frac{gr}{r}$ (or simply g,) and the value thus ascertained for F would be the velocity that it would generate in a body, acting uniformly for 1' on the body. Thus suppose a body revolving uniformly in circle P E D with such a velocity as it would acquire by a perpen- dicular fall down line M B here on earth's surface; this datum gives the exact velocity in the orbit, be- cause in perpendicular falls by gravity $V^2 \propto$ space fallen through, and
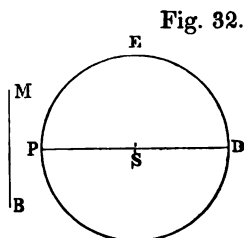
Fig. 32.

$$\therefore \frac{y}{2} : M\,B :: g^2 : V^2 = 2\,g\ .\ M\,B \therefore F : \text{gravity}$$

$$:: \frac{2\,M\,B\ .\ g}{P\,S} : g :: 2\,M\,B : P\,S.$$

If the orbit P E D (fig. 32.) be uniformly de-scribed by a small weight attached to the end of a string, and the string be whirled round by the hand, so as to preserve the velocity $\sqrt{2\,g\ .\ M\,B}$ uniformly in the orbit, 2 M B : P S gives the ratio of tension of the string P S, (produced by its re-action in resisting the centrifugal effort of the weight,) to the tension produced in the string by the same weight hanging freely and quietly. In any circular orbit, where V is uniform, the centri-fugal force of the revolving body is exactly equal to the centripetal force exerted on the body. If the velocity of a revolving body is increased, the centripetal force must be increased, as the body now makes a more powerful effort against being drawn out of a rectilinear direction. And on the other hand, if our earth had been placed nearer to the sun, a greater projectile velocity must have been given to it, as a balance against the more powerful attraction of the sun. The formula of Prop. IV. $F \propto \frac{V^2}{R}$ shews that centrifugal tendencies of any revolving bodies or particles of matter increase with increase of their velocity, and if their velocity remain the same, they increase with decrease of R. This decrease of R increases the curvature of the circle, in which they are constrained to move ; and consequently more violence is put on their natural propensity to move in a straight line.

In wheels, and in many parts of machinery, the centrifugal tendencies need to be provided against; and the force counteracting these tendencies often consists in the cohesion of particles of matter, or in some reaction of certain parts. The bulging, or oblate form of the earth and some other planets in their equatorial regions, is caused by the greater velocity of the motion of the particles in those regions of a planet, over the motion of particles in the polar regions, in the revolving of the planet round its axis.

PROP. V. *Given the velocities of a body, and the directions of its motion at three points of its orbit, to determine the position of the centre of force.*

Let P M, M Q N, N R be the directions, in which the body is moving at the three points P, Q, R; draw P p, Q q, R r at right angles to these lines respectively, and such that

$$P p : Q q = \text{velocity at Q : velocity at P,}$$
$$\text{and } Q q : R r = \text{velocity at R : velocity at Q.}$$

Through p, q, r draw $p\,m$, $m\,n$, $n\,r$ respectively parallel to PM, MN, NR; join M m, N n, and produce them to meet in S; S will be the centre of force.
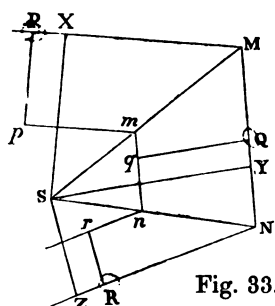
Fig. 33.

Draw SX, SY, SZ, perpendicular to P M, M N, N R, respectively,

$$\text{then } \frac{SX}{SM} = \frac{Pp}{Mm},$$

$$\text{and } \frac{SM}{SY} = \frac{Mm}{Qq};$$

$$\therefore \frac{SX}{SY} = \frac{Pp}{Qq} = \frac{\text{velocity at Q}}{\text{velocity at P}}.$$

$$\text{Similarly } \frac{SY}{SZ} = \frac{\text{velocity at R}}{\text{velocity at Q}}.$$

Hence the perpendiculars, drawn from S upon the tangents at P, Q, R, are inversely as the velocities at those points; therefore S must be the centre of force. (Prop. I. Cor. 1.)

PROP. VI. *If a body in a space void of resistance revolves in any orbit about an immoveable centre, and in an indefinitely small time describes any nascent arc; and the sagit of that arc is supposed to be drawn, which may bisect the chord, and being produced may pass through the centre of force; the centripetal force, in the middle of the arc, will be as the sagit directly, and the square of the time inversely.*

For the sagit, in a given time, is as the force (by Cor. 4. Prop. I.) and increasing the time in any ratio, because the arc will be increased in the same ratio, the sagit will be increased in the duplicate of that ratio (by Cor. 2. and 3. Lemma XI.); and therefore is as the force, and the square of the time. Subduct on both sides the duplicate ratio of the time, and the force will be as the sagit directly, and the square of the time inversely. Which was to be demonstrated.

To understand this demonstration, let K P L, bpq, be two indefinitely small arcs, described in the same time, t, and . P N, pc, be their sagits which bisect their chords, and go down to their respective centres of force, S and s; then by Prop. I.
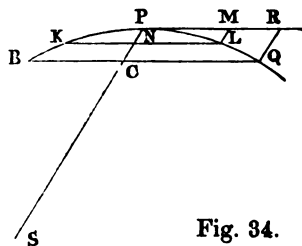
Fig. 34.

Cor. 4. P N $: pc ::$ force at P : force at p. Now let arc K P L be enlarged to arc BPQ, requiring time T for its description, and having P C for its sagit; then by Cor. 2. Lemma XI. P C : P N :: P Q^2 : P L^2 :: T^2 : t^2 compound these two proportions,
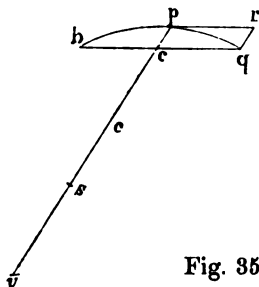
Fig. 35.

$$P N : pc :: F . \text{ at } P : f . \text{ at } p$$
$$P C : P N :: T^2 : t^2$$
$$\overline{P C : pc :: F . T^2 : f . t^2.}$$

Subduct on both sides, Newton says, the duplicate ratio of the time; i. e. divide the 1st and 3d of the four proportionals by T^2, and the 2d and 4th by t^2, and then F $: f :: \frac{PC}{T^2} : \frac{pc}{t^2}$. As the arcs (B P Q and K P L) are indefinitely small, they may be supposed to be described with uniform velocities, and therefore to be as T : t, the times of their description; and this is assumed the ratio of

the half arcs. Henceforward we will in our figures and reasonings take, as Evans does, the half arcs P Q, and the subtenses to angle of contact Q R, drawn parallel to line from P to S, and equal to the proper sagit of the arc, double of P Q.

The above general ratio or value of the centripetal force was proved in Cor. Lemma X. R Q is ultimately the space, through which F draws the body from P R its rectilinear direction at P in time T; and $r\,q$ is ultimately the space, through which another constant force f draws another body from $p\,r$ its rectilinear direction at p in time t. If these two constant forces might be considered as acting uniformly, they would be to one another as $\dfrac{V}{T} : \dfrac{v}{t}$, V and v being the velocities generated by their respective action in times T and t, since in motion uniformly accelerated $V \propto T$. Now although F and f may not be acting uniformly, yet by Lemma X. ipso motûs initio, we may consider the space through which F carries a body to vary in the duplicate ratio of T, the time of its description; and this is in fact bringing the action of F to uniform action, where we take T in its limit, i. e. indefinitely small. And the same holds with regard to the other constant force, f. Therefore Newton, specifying the arcs as nascent, described in indefinitely small times, T and t, we may consider the action of F and f uniform through those times, and $\therefore$ F : f $:: \dfrac{V}{T} : \dfrac{v}{t} :: \dfrac{V \cdot T}{T^2} : \dfrac{v \cdot t}{t^2}$. But V . T gives a

space double the space, through which F has drawn the body in time T; and the same holds with regard to $v \cdot t$. But the subtenses R Q and $r\,q$ are ultimately these spaces, through which F and f draw the bodies in the times T and t, $\therefore$ F $: f :: $ L. R. $\frac{2\,RQ}{T^2} : \frac{2\,r q}{t^2}$. Evans and Whewell, adopting equations (and not proportions, as Newton does,) in their reasoning make F $= 2$ limit $\frac{QR}{T^2}$.

As our demonstration in this Prop. does not suppose the two bodies in the same orbit, or even in orbits round the same centre of force, our conclusion will apply either to different points in the same orbits, or to points in different orbits, whether round the same or different centres. But in the three first Corollaries of this Prop. Newton substitutes for the duplicate ratio of T : t, the duplicate ratio of the areas, described in those times. This limits the conclusions of these Corollaries to different points in the same orbit, as only in the same orbit are areas necessarily proportional to the times of their description.

Cor. 1. If a body P, revolving about the centre S, describes a curve line A P Q, and a right line Z P R touches that curve in any point P; and, from any other point Q of the curve, Q R is drawn parallel to the distance S P, meeting the tangent in R; and Q T is drawn perpendicular to the distance S P; the centripetal force will be reci-

procally as the solid $\dfrac{SP^2 \times QT^2}{QR}$; if the solid is
taken of that magnitude which it ultimately ac-
quires, sup-
posing the
points　P
and Q con-
tinually to
approach
to each
other. For
QR is equal
to the sagit

Fig. 36.

of double the arc Q P, in whose middle is P; and
double the triangle S Q P, or S P × Q T is pro-
portional to the time, in which that double arc is
described; and therefore may be used for the
exponent of the time.

Cor. 2. By a like reasoning the centripetal force
is reciprocally as the solid $\dfrac{SY^2 \times QP^2}{QR}$; if S Y is a
perpendicular, let fall from the centre of force on
P R, the tangent of the orbit. For the rectangle
S Y × Q P and S P × Q T are equal.

Cor. 3. If the orbit is either a circle, or touches
or cuts a circle concentrically, that is, contains with
a circle an indefinitely small angle of contact or
section, having the same curvature and the same
radius of curvature at the point P; and if P V is a
chord of this circle, drawn from the body through
the centre of force; the centripetal force will be

reciprocally as the solid $S Y^2 \times P V$. For $P V$ $= \frac{P Q^2}{Q R}$.

The rectangles under $S P . Q T$ and $S Y . Q P$ are each double of the triangle $P S Q$, which (Lemma VIII.) is equal to the area described in time T. Take, in the same orbit, another arc $p q$, indefinitely small, described in time t, and draw the corresponding lines for this arc $p q$, viz. $S p$. $S q$, $q t . p r$, $q r . S y$: then the rectangles under $S p . q t$, or under $S y . q p$ are each double of the triangle $S p q$, the area described in time t. That $P V$, the chord of curvature, $= \frac{Q P^2}{Q R}$. See Prop. III. Section I.

Cor. 4. The same thing being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular S Y, by Cor. 1. Prop. I.

Obs. 1. The 4th Cor. thus deduced, is applicable only to different points in the same orbit. But it may be deduced at once from Prop. VI. and thus give a ratio of F to f of application, as unlimited as the ratio of the Prop. The indefinitely small arc P Q in Fig. 34. of Prop. VI. may be considered as described (in time T) with V, the velocity at P continued uniform $\therefore$ P Q = V. T. and $\therefore T^2 = \frac{P Q^2}{V^2}$, for the same reason the arc $p q = v.t.$ and $\therefore t^2 = \frac{p q^2}{v^2}$.

Substitute these values of $T^2 \propto t^2$ in the formula established by Prop. VI. and then you have $F : f$ $:: \frac{Q R}{P Q^2} . V^2 : \frac{q r}{p q^2} . v^2$; but $\frac{P Q^2}{Q R}$ and $\frac{p q^2}{q r}$ are equal to the chords of curvature $P V$ and $p v$, at points P and p, drawn parallel to the subtenses $Q R$ and $q r$. The velocities V and v, here spoken of, are the velocities which the revolving bodies have at P and p, and must not be confounded with the velocities specified in the demonstration of the Prop. as produced by the action of F and f, pulling the bodies from their tangential directions at P and p.

Obs. 2. Another value for T and t may be found, capable of being incorporated with the expressions for F and f in the three first Corollaries, making those expressions general. Thus (Fig. 36.) $T : 1'$ $::$ Triangle $P S Q$: area described in $1' \therefore T =$ $\frac{P S Q}{\text{area in } 1'} = \frac{S P . Q T}{2 \text{ area in } 1'}$, in the same manner $t =$ $\frac{\text{triangle } p s q}{\text{area in } 1' \text{ of the orbit of the arc } p q} = \frac{2 p s q, \text{ or } s p . q t}{2 \text{ area in } 1' \text{ of that orbit}}$

$\left\{ \begin{array}{l} \therefore \text{ putting A for area described in} \\ \quad 1' \text{ in orbit of arc } P Q \\ \text{and } a \text{ for area of } 1' \text{ in the other} \\ \quad \text{orbit of } p q \end{array} \right\}$ we have $T^2 : t^2 :: \frac{S P^2 . Q T^2}{4 A^2}$ $: \frac{s p^2 . q t^2}{4 a^2}$.

Thus the centripetal force is reciprocally as $\frac{S Y^2 . Q P^2}{4 A^2 . Q R}$ and as $\frac{S P^2 . Q T^2}{4 A^2 . Q R}$ and as $\frac{S Y^2 . P V}{4 A^2}$.

Cor. 5. The space $p o$, through which a body must descend from rest by the action of the

centripetal force at p (Fig. 35.) continued uniform, in order to acquire the velocity it has in the curve at $p = $ to $\frac{1}{4}$ of the chord of curvature $p\,v$. In the very small time t, while the body moves from p to q, the centripetal force draws it through $r\,q$, and gives it such a velocity as would, if continued uniform, carry it, in time t, through space $= 2\,r\,q$. Therefore, since the velocities are as the spaces uniformly described in the same time, vel. in curve at p : vel. acquired through $r\,q :: p\,q : 2\,r\,q \therefore V^2$ in curve : v^2 through $r\,q : : p\,q^2 : 4\,r\,q^2$, but v^2 through $r\,q : V^2$ through $p\,o$ (i. e. V^2 in the curve) $:: r\,q : p\,o$.

Compound these proportionals : product of the two first $=$ product of the two second, $\therefore$ product of the two third $=$ product of the two fourth, $\therefore p\,q^2 \cdot r\,q = 4\,r\,q^2 \cdot p\,o, \therefore \frac{p\,q^2}{4\,r\,q} = p\,o$, thus $p\,o = \frac{1}{4}$ of the chord of curvature.

On the ground that a constant force is represented by the velocity which its uniform action for $1'$ would generate, we might, in accordance with the suppositions stated in our demonstrations for Prop. VI. assume $F = 2$ limit $\frac{Q\,R}{T^2}$, as Evans and Whewell have done. And then substituting certain values of T^2, we should obtain equations for F, corresponding to the proportions which the four Corollaries in Newton establish for $F : f$. Thus, putting $h = 2$ area described in $1'$, $T' : 1'$ $:: 2$ area $P\,S\,Q : h \therefore T = $ limit $\frac{Q\,T \cdot S\,P}{h}$ also.

$$T = \frac{\text{limit } Q\,P\,.\,S\,Y}{h}, \text{ hence } F = 2 \lim. \frac{Q\,R}{T^2} = \frac{2\,h^2}{S\,P^2} \cdot$$

$\text{limit } \dfrac{Q\,R}{Q\,T^2} \text{ also} = \dfrac{2\,h^2}{S\,Y^2}. \text{ limit } \dfrac{Q\,R}{Q\,P^2}, \text{ but limit } \dfrac{Q\,R}{Q\,P^2} =$

$\dfrac{1}{P\,V} \;\therefore\; F = \dfrac{2\,h^2}{S\,Y^2\,.\,P\,V}.$ Again, in our proof of

Cor. 4, we see that ultimately $T^2 = \dfrac{P\,Q^2}{V^2} \;\therefore\; 2 \text{ limit}$

$\dfrac{Q\,R}{T^2} = 2\,V^2 \lim. \dfrac{Q\,R}{P\,Q^2} = \dfrac{2\,V^2}{P\,V}.$ Thus $F = \dfrac{2\,V^2}{P\,V}$ and

$\therefore\; V^2 = F\,.\,\dfrac{P\,V}{2}.$ Now let S be the space due to

V by the action of F continued constant, i. e. the space through which F acting uniformly must carry a body from rest, to generate in it a velocity = velocity which a body moving in the curve has at P.   Then $V^2 = 2\,F\,S$, because by principles of uniformly accelerated motion (that $S \propto V^2$), and of the notation for F, (that it is expressed by the velocity generates in $1'$,) $\dfrac{F}{2} : S :: F^2 : V^2$.   Equate

these two values of V^2, $\therefore\; F\,.\,\dfrac{P\,V}{2} = 2\,F\,S \;\therefore\; S =$

$\dfrac{P\,V}{4}$, i. e. $\frac{1}{4}$ of the chord of curvature at P is the space through which F must draw a body from rest, to give it the velocity which it has revolving in the curve at P.   Where the body revolves

uniformly in the circle, $\dfrac{P\,V}{4} = \dfrac{R}{2}$ and $V^2 = 2\,F\,S$

$= \dfrac{F}{2}\,.\,2\,R$; thus V, or the circular arc of $1'$ is the mean proportional between the space fallen through by F in $1'$ and the diameter of the circle.   Also

$v^2\,t^2 = \dfrac{F}{2}\,t^2\,2\,R$; but $\dfrac{F}{2}\,t^2 =$ space described from

rest by F in t'', because $1' : t^2 :: \dfrac{F}{2} : S = \dfrac{F}{2}\,t^2.$

Cor. 6. If V, *v* be the velocities at P, *p*, points similarly situated in similar orbits, described round S, *s* centres of force, also similarly situated,

Force at P (F) : force at *p* $(f) = \dfrac{V^2}{SP} : \dfrac{v^2}{sp}.$

Let P Q, *p q* be similar arcs of the similar orbits; Q R, *q r* subtenses of these arcs parallel to S P, *s p*, and P V, *p v* chords of curvature at P, *p*,
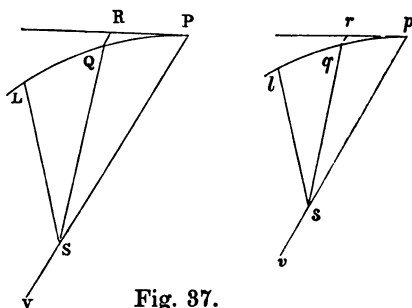
Fig. 37.

through S, *s*. Then by Cor. 4. Prop. VI. F at P : *f*, at *p* :: $\dfrac{V^2}{PV} : \dfrac{v^2}{pv}$.

Now since P $\propto$ *p* are points similarly situated in similar orbits round S and *s*, centres of force, also similarly situated, P V : *p v* :: S P : *s p*. For P Q and *p q* being similar arcs, by similar triangles, $\dfrac{PQ}{QR} = \dfrac{pq}{qr}.$ But $\dfrac{PV}{PQ} = \dfrac{PQ}{QR}$ because $PV = \dfrac{PQ^2}{QR}$, and for the same reason $\dfrac{pv}{pq} = \dfrac{pq}{qr} \therefore \dfrac{PV}{PQ} = \dfrac{pv}{pq}$ $\therefore$ P V : *p v* :: P Q : *p q*, and by similar triangles, S P Q, *s p q*, P Q : *p q* :: S P : *s p* $\therefore$ F : *f* :: $\dfrac{V^2}{SP}$: $\dfrac{v^2}{sp}.$

Again, let T and *t* be times of describing the similar evanescent arcs P Q and *p q*, then V and *v* may be

supposed unchanged during the description of these arcs; $\therefore$ $V^2 : v^2 :: \dfrac{P Q^2}{T^2} : \dfrac{p q^2}{t^2}$ $\therefore$ F : f $\left(\dfrac{V^2}{S P} : \dfrac{v^2}{s p}\right) :: \dfrac{P Q^2}{S P . T^2} : \dfrac{p q^2}{s p . t^2}$, but P Q : $p q$ :: S P : $s p$ $\therefore$ F : $f \dfrac{S P^2}{S P . T^2} : \dfrac{s p^2}{s p . t^2} :: \dfrac{S P}{T^2} : \dfrac{s p}{t^2}$. Now since S P Q and $s p q$ are similar areas of similar figures (let us suppose similar ellipses), T and t the times of their description will be to one another as P and p, the periodical times of the whole similar orbits.   For the first Prop. Section II.

$$\left.\begin{array}{l} \text{P : T :: whole ellipse : P S Q} \\ p : t \text{ :: whole ellipse : } p s q \end{array}\right\} \begin{array}{l} \text{but by similar figures, larger} \\ \text{ellipse : P S Q as smaller} \\ \text{ellipse : } p s q = \therefore \text{P : } p :: \\ \text{T : } t. \end{array}$$

Thus in the two orbits, supposed in the eighth Cor. Prop. IV. we have shewn that at any two similar points F : f :: $\dfrac{V^2}{S P} : \dfrac{v^2}{s p}$, the velocities and distances of those two points; also that F : f :: as the distances of the two similar points directly, and the periodical times of the entire orbits inversely. And these were the two principles established in the two first Corollaries of Prop. IV. from which the following four Corollaries follow as necessary consequences.   If we had begun this case with taking P X and $p x$, two evanescent simultaneous arcs, and X Y . $x y$ the subtenses, then we should have had, without reference to Prop. VI. F : f :: X Y : $x y$ :: $\dfrac{P X^2}{P V} : \dfrac{p x^2}{p v} :: \dfrac{V^2}{P V} : \dfrac{v^2}{p v}$. Thus this Corol-

lary may be proved as the eighth Cor. of Prop. IV, as the remaining part of our demonstration (P Q and $p\ q$ being taken similar, not simultaneous, evanescent arcs) wants no aid from Prop. VI. Compare the case of the circles supposed in Prop. IV. with the figures, say two similar ellipses, supposed in Cor. 8. In both cases, time of evanescent arcs being given, F $\propto$ subtense, and in both cases subtense $= \frac{\text{arc}^2}{\text{P V}}$, and arc in both cases $\propto$ V, and P V in one case $= 2$ R, in the other case P V $\propto$ S P the distance. In the circles, V being uniform $\propto \frac{\text{R}}{\text{P}}$. In the ellipses, V $\propto \frac{\text{P Q}}{\text{T}}$, and P Q $\propto$ S P, and by equable description of areas T $\propto$ P.

Cor. 7. Hence, if any curvilinear figure A P Q is given, (see Fig. 36.) and therein a point S is also given, to which a centripetal force is perpetually directed; the law of centripetal force may be found, by which the body P, continually drawn back from a rectilinear course, will be retained in the perimeter of that figure, and will describe the same by a perpetual revolution. That is, we are to find by computation, either the solid $\frac{\text{S P}^2 \times \text{Q T}^2}{\text{Q R}}$, or the solid S Y^2 × P V, reciprocally proportional to this force. Examples of this we shall give in the following Problems.

PROP. VII. PROBLEM II. *Let a body revolve in the circumference of a circle ; it is required to find the law of centripetal force tending to any given point.*

Let V Q P A be the circumference of the circle ; S the given point, to which the force tends, as to a centre ; P the body moving in the circumference ; Q the next place into which it is to move, and P R Z the tangent of the circle at the preceding place. Through the point

Fig. 38.
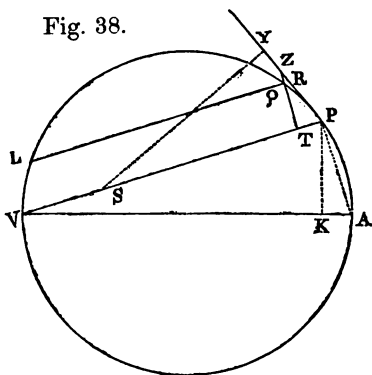
S let the chord P V be drawn ; and, the diameter V A of the circle being drawn, let A P be joined ; and let fall Q T perpendicular to S P, which produced may meet the tangent P R in Z ; and lastly, through the point Q let L R be drawn, which may be parallel to S P, and may both meet the circle in L, and the tangent P Z in R. And, because of the similar triangles Z Q R, Z T P, V P A, R P², that is Q R L will be to Q T², as A V² to P V². And, therefore, $\frac{Q\,R\,L \times P\,V^2}{A\,V^2}$, is equal to Q T². Let these equals be multiplied into $\frac{S\,P^2}{Q\,R}$, and the points P and Q continually approaching, for R L write P V. Thus we shall find $\frac{S\,P^2 \times P\,V^3}{A\,V^2} = \frac{S\,P^2 \times Q\,T^2}{Q\,R}$. There-

fore, (by Cor. 1 and 5, Proposition VI.) the centri-petal force is reciprocally as $\frac{S\,P_2 \times P\,V_3}{A\,V_2}$; that is (because $A\,V^2$ is given) reciprocally as the square of the distance or altitude S P, and the cube of the chord P V jointly. Which was to be found.

The solid $S\,Y^2\,.\,P\,V$, the reciprocal of F, (Cor. 3. Prop. VI.) does very briefly give the same expression for the variation of F. S Y, the perpendicular from S to the tangent at P, being drawn, the triangles S P Y and V P A are similar, $\therefore$ S Y : S P $: :$ V P : A V, $\therefore$ S Y $= \frac{S\,P\,.\,V\,P}{A\,V}$, $\therefore$ S Y^2 . P V $= \frac{S\,P^2\,.\,P\,V^3}{A\,V^2}$.

These expressions are limited to the variation of F at different points in the same orbit; to make them applicable to comparing of centripetal forces in different circular orbits, whether round the same or different centres of force, we see that A V the diameter is no longer the same in the different orbits, and that $4\,A^2$, i. e. four times the area dato tempore of each orbit, must be incorporated into the original values of the centripetal forces.

Cor. 1. Hence, if the given point S, to which the centripetal force always tends, is placed in the circumference of this circle, suppose at V, the centripetal force will be reciprocally as the qua-drato-cube (or fifth power) of the altitude S P.

Cor. 2. The force by which the body P in the circle A P T V revolves about the centre of force S, is to the force by which the same body P may revolve in the same circle, and in the same periodical time, about any other centre of force R, as

Fig. 39.

R P² × S P, to the cube of the right line S G, which is drawn from the first centre of force S, to the tangent of the orbit P G, and is parallel to the distance P R of the body from the second centre of force R.   Also, if S Y and R Y were drawn from S and R perpendicular upon the tangent at P, they would, by similar triangles, be to one another as S G to R P, ∴ velocity at P, round S, is to velocity at P, round R, as R P to S G.

For, by the construction of this Proposition, the former force is to the latter, as R P² × P T³ to S P² × P V³; that is, as S P × R P² to $\frac{S P^3 \times P V^3}{P T^3}$; or (because of the similar triangles P S G, T P V) to S G².

Suppose the two cases here stated to be two distinct orbits; in the two orbits here supposed, the diameters are the same, and A, area dato tempore is the same, ∴ F : ƒ (at the point P)

simply as $R\,P^2\,.\,P\,T^3 : S\,P^2 : P\,V^3$. If the periodical time were not the same in the two orbits, then $F : f$, at the same point $P :: \dfrac{R\,P^2\,.\,P\,T^3}{A^2}$ $: \dfrac{S\,P^2\,.\,P\,V^3}{B^2}$, A and B being the areas dato tempore about S and about R, $\therefore F : f :: \dfrac{S\,P\,.\,R\,P^2}{A^2} : \dfrac{S\,G^3}{B^2}$. If P and p be the periodical times, $P\,.\,A = p\,B \therefore A : B :: p : B.$

Cor. 3. The force, by which the body P in any orbit revolves about the centre of force S, is to the force, by which the same body P may revolve in the same orbit, and in the same periodical time, about any other centre of force R, as the solid $S\,P \times R\,P^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line S G, which is drawn from the first centre of force S to the tangent P G of the orbit, and is parallel to the distance R P of the body from the second centre of force R. For the forces in this orbit, at any point P, are the same as in a circle of the same curvature.

Suppose an orbit, not circular, to touch P G (Fig. 39.) at P, and at P to have the same curvature with the circle P T V. Suppose a body to revolve in this orbit with centripetal force, directed to S. Again, suppose a body to be revolving in the osculating circle, with the same centripetal force directed to the same point S. These two orbits

at P would have the evanescent arc coinciding, and the same subténse, and the same velocity, since F and P V would be the same. Hence the forces acting on the two bodies at P are the same, and in the same directions. Repeat this supposition, making R the centre of force for the two bodies revolving as before, and calculating what must influence each body on their coming to P. Therefore, whatever is proved in the last Corollary, as true for the body in the circle, when coming to P, and in one case having S, in another case having R for the centre of force, will be true also for the body in the other orbit on coming to P, under similar circumstances. From the value of S Y, velocity at different points in the same circle $\propto \dfrac{1}{S\,P\,.\,P\,V}$. Applying the general expression for the variation of centripetal force obtained in this Prop. to the circular orbits of Prop. IV, $F \propto \dfrac{A\,V^2\,.\,4\,A^2}{S\,P^2\,.\,P\,V^3}$, we see that in the same orbit F is constant, and that in different orbits $F \propto \dfrac{4\,A^2}{S\,P^2\,.\,P\,V}$ ∴ in those cases when $F \propto \dfrac{1}{R^2}, \dfrac{4\,A^2}{2\,R}$ is invariable; hence $A^2 : a^2 =$ R . r. This accords with Prop. XIV.

PROP. VIII. PROBLEM III. *Let a body move in the semi-circum- ference* P Q A; *it is required to find the law of centripetal force, tending to a point* S, *so remote, that all lines* P S, R S *drawn thereto, may be taken for parallel.*

From C, the centre of the semi-circle, let the semi-diameter C A be drawn, cutting those parallels

perpendicularly in M and N, and let C P be joined.
Because of the simi-
lar triangles C P M,
P Z T, and R Z Q,
C P² is to P M², as
P R² to Q T²; and,
from the nature of
the circle, P R² is
equal to the rect-

angle Q R × $\overline{R N + Q N}$; or, the points P and
Q continually approaching, to the rectangle Q R
× 2 P M. Therefore C P² is to P M², as Q R ×
2 P M to Q T²; therefore $\frac{Q T^2}{Q R} = \frac{2 P M^3}{C P^2}$, and
$\frac{Q T^2 \times S P^2}{Q R} = \frac{2 P M^3 \times S P^2}{C P^2}$. And therefore (by
Cor. 1 and 7. Prop. VI.) the centripetal force is
reciprocally as $\frac{2 P M^3 \times S P^2}{C P^2}$; that is, (neglecting
the given ratio $\frac{2 S P^2}{C P^2}$) reciprocally as P M³. Which
was to be found.
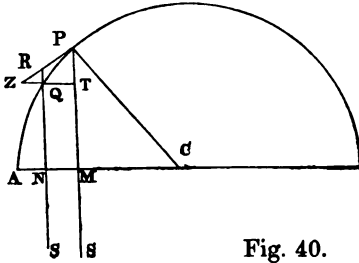
The same thing is likewise easily collected from
the preceding Proposition.

In the case before us, P V = 2 P M, and S being
infinitely distant, the lines from P, Q, &c. to S may
be regarded as equal, since their difference will be
indefinitely small compared with them. By Prop.
VII. F $\propto \frac{A V^2}{S P^2 . P V^3}$. A V, the diameter, is in-
variable for the several points of the same semi-
circumference, and S P may be regarded as given,
∴ F $\propto \frac{1}{P V^3} \propto \frac{1}{2 P M^3}$. The same conclusion is
also briefly obtained by working, as in the last

Prop., by the formula[*] $F \propto \dfrac{1}{S\,Y^2\,.\,P\,V}$. The re-
moteness of S will not hinder the triangles S Y P
and P V A being similar. The lines S Y and S P
are infinitely great, compared with the finite lines
P V and A V, but S Y has to S P a finite ratio, the
ratio P V : A V. If we resolve the motion, sup-
posed in this case, viz. P Q in time T, into a per-
pendicular motion towards S, viz. P T, and a
horizontal motion T Q, we shall find the horizontal
velocity constant; A being area described round S
in 1′, S P . Q T : 2 A :: T : 1, by the first Prop.
Section II. $\therefore \dfrac{1}{T} = \dfrac{2\,A}{S\,P\,.\,Q\,T} \therefore \dfrac{Q\,T}{T} = \dfrac{2\,A}{S\,P}$, a con-
stant quantity in the case before us. Thus T Q
the horizontal vel. $\propto$ T. The velocity at any
point $\propto$ inversely as S Y, the perpendicular from S
upon tangent to point P ; and $S\,Y \propto S\,P\,.\,P\,V$,
$\therefore$ vel. $\propto \dfrac{1}{P\,V} \propto \dfrac{1}{P\,M}$.

SCHOLIUM TO PROP. VIII.

*If A Q P be any conic section, it may be described by the action of
a force tending to a point at an infinite distance, and varying
inversely as the cube of the ordinate.*

Let P O, the diameter of
curvature at P, cut the axis of
the conic section in K ; draw
O V perpendicular to P S, then
P V is the chord of curvature
at P in direction of the force ;
and complete the construction
as in the proposition.
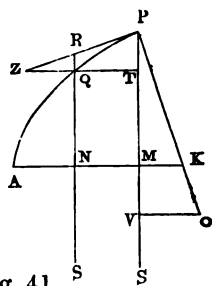
Fig. 41.

By similar triangles $Z\,P\,T$, $P\,M\,K$,

$$\frac{Q\,T^2}{Q\,R} : \frac{R\,P^2}{Q\,R} = Z\,T^2 : Z\,P^2 = P\,M^2 : P\,K^2,$$

and this being true always, will be true when Q moves up to P,

$$\therefore \text{L. R.} \ \frac{Q\,T^2}{Q\,R} : \frac{R\,P^2}{Q\,R} = P\,M^2 : P\,K^2,$$

$$\text{and } P\,V : P\,O = P\,M : P\,K,$$

$$\therefore \text{since limit } \frac{R\,P^2}{Q\,R} = \text{limit } \frac{P\,Q^2}{Q\,R} = P\,V,$$

$$\text{limit } \frac{Q\,T^2}{Q\,R} : P\,O = P\,M^2 : P\,K^2.$$

Now in all conic sections, the diameter of curvature $= \frac{8}{L^2} \cdot P\,K^2$,

$$\therefore \text{ limit } \frac{Q\,T^2}{Q\,R} = \frac{8\,P\,M^3}{L^2},$$

$$\text{and } \therefore F \propto \frac{L^2}{3\,S\,P^2 \cdot P\,M^3} \propto \frac{1}{P\,M^3}.$$

1. To find the chords of curvature through the centre and focus, and the diameter of curvature, at any point of an ellipse and hyperbola. Vide Figs. 46 and 47, (Props. XI. and XII.)

Let $Q\,v$, a semi-ordinate to the diameter $P\,C\,G$, cut $S\,P$ in x, $C\,P$ in v, and $P\,F$, which is perpendicular to the semi-conjugate $C\,D$, in u.

Chord of curvature through C

$$= \text{limit } \frac{P\,Q^2}{\text{subtense parallel to C P}}$$

$$= \text{limit } \frac{P\,Q^2}{P\,v} = \text{limit } \frac{Q\,v^2}{P\,v}$$

$$= \text{limit } \frac{C\,D^2}{C\,P^2} \cdot v\,G, \text{ since } \frac{Q\,v^2}{P\,v \cdot v\,G} = \frac{C\,D^2}{C\,P^2}$$

$$= \frac{2\,C\,D^2}{C\,P}, \text{ since } v\,G \text{ ultimately} = 2\,C\,P.$$

Chord of curvature through S

$$= \text{limit } \frac{P\,Q^2}{\text{subtense parallel to S P}} = \text{limit } \frac{Q\,v^2}{P\,x}$$

$$= \text{limit } \frac{Q\,v^2}{P\,v} \cdot \frac{P\,v}{P\,x} = \text{limit } \frac{C\,D^2}{C\,P^2} \cdot v\,G \cdot \frac{P\,C}{P\,E}$$

$$= \frac{2\,C\,D^2}{A\,C}, \text{ since } P\,E = A\,C.$$

Diameter of curvature

$$= \text{limit } \frac{P\,Q^2}{\text{subtense perpendicular to tangent}}$$

$$= \text{limit } \frac{Q\,v^2}{P\,u} = \text{limit } \frac{Q\,v^2}{P\,v} \cdot \frac{P\,v}{P\,u} = \text{limit } \frac{C\,D^2}{C\,P^2} \cdot v\,G \cdot \frac{P\,C}{P\,F}$$

$$= \frac{2\,C\,D^2}{P\,F}.$$

Cor. Let PF cut the axis major in K, then $PF . PK = BC^2$;

also $CD . PF = AC . BC$,

$\therefore$ diameter of curvature $= \dfrac{2AC^2 . BC^2}{PF^3} = \dfrac{2AC^2 . BC^2 . PK^3}{BC^6}$

(but L is third proportional to $2AC$ and $2BC$

$$\therefore L^2 = \frac{4BC^4}{AC^2}) = \frac{8PK^3}{L^2}.$$

2. To find the chord of curvature through the focus, and the diameter of curvature at any point of a parabola.   Vide Fig. 48, (Prop. XIII.)

Let QV, a semi-ordinate to the diameter PV, cut SP in X; and the normal PK in U, draw SY perpendicular to the tangent at P; then $PX = PV$, hence,

Chord of curvature through S

$= \text{limit} \dfrac{PQ^2}{\text{subtense parallel to } SP}$

$= \text{limit} \dfrac{PQ^2}{PX} = \text{limit} \dfrac{QV^2}{PV}$

$= 4SP$, since $QV^2 = 4SP . PV.$

Diameter of curvature

$= \text{limit} \dfrac{PQ^2}{\text{subtense perpendicular to tangent}} = \text{limit} \dfrac{QV^2}{PU}$

$= \dfrac{4SP . PV}{PU} = 4SP . \dfrac{SP}{SY}$, by sim^r. triangles PVU, SPY,

$= \dfrac{4SP^2}{SY}$, or $= 4 \sqrt{\dfrac{SP^3}{SA}}.$

Cor. Let P Y meet the axis in T, then

$$S T = S P = S K, \therefore S Y = \tfrac{1}{2} P K;$$

hence diameter of curvature

$$= 4 \frac{S P^2}{S Y} = 4 \frac{S Y^4}{S A^2 . S Y} = \frac{4}{S A^2} S Y^3 = \frac{1}{2 S A^2} . P K^3$$

$$= \frac{8 P K^3}{L^2}.$$

PROP. IX. PROBLEM IV. *Let a body revolve in a spiral* P Q S, *cutting all the radii* S P, S Q, &c. *in a given angle; it is required to find the law of centripetal force, tending to the centre of that spiral.*

Let the indefinitely small angle P S Q be given; and because all the angles are given, the species of

Fig. 42.

the figure S P R Q T will be given. Therefore the ratio $\frac{Q T}{Q R}$ is given; and $\frac{Q T^2}{Q R}$ is as Q T; that is, (because the species of that figure is given,) as SP.

But if the angle PSQ is any way changed, the right line QR, subtending the angle of contact QPR (Lemma XI.) will be changed in the duplicate ratio of PR or QT. Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before; that is, as SP. Therefore $\frac{QT^2 \times SP^2}{QR}$ is as PS^3, and (by Cor. 1. and 7. Prop. VI.) the centripetal force is reciprocally as the cube of the distance SP. Which was to be found.

This demonstration will be satisfactory, when developed, as follows, under its two separate cases.

Case 1. The spiral is supposed to cut all the radii SP, SQ, Sp, &c. at a given angle. Therefore, if the $\angle$ PSQ, pSq, are supposed indefinitely small and equal, the two figures $SPRQT$, $Sprqt$, are similar: hence $QT : QR :: qt : qr$, or $\frac{QT}{QR} = \frac{qt}{qr} \therefore \frac{QT}{QR} \times QT$ $\left(\text{or } \frac{QT^2}{QR}\right) : \frac{qt}{qr} \times qt$ $\left(\text{or } \frac{qt^2}{qr}\right) :: QT : qt :: SP : Sp$.

Case 2. Now take $\angle pSM$ no longer equal $\angle PSQ$; $ML : qr :: Mp^2 : qp^2$ by Cor. 1. Lemma XI. but NMp and tqp are similar $\triangle^s$, $\therefore Mp : qp :: MN$ $: qt \therefore ML : qr :: MN^2 : qt^2 \therefore \frac{MN^2}{ML} = \frac{qt^2}{qr}$; therefore even when the angle psq is in any way changed, still $\frac{QT^2}{QR} : \frac{MN^2}{ML} \left(= \frac{qt^2}{qr}\right) :: SP : Sp$

$\times$ the 1st and 3d proportionals by $S P^2$, the 2d and 4th by $S p^2$, and $\dfrac{Q T^2 \times S P^2}{Q R}$ $\left(\text{or } \dfrac{1}{F}\right)$: $\dfrac{M N^2 \times S p^2}{M L}$ $\left(\text{or } \dfrac{1}{f}\right)$: : $S P^3 . S p^3$.

The formula, $F \propto \dfrac{1}{S Y^2 . P V}$, would easily lead to the same conclusion. Suppose the perpendiculars S Y and S y to be drawn to the tangents P R and pr. The triangles SYP, Syp, are similar,

Fig. 43.

and $\therefore$ S Y : S y :: S P : S p. Also from the properties of this curve $P V = 2 S P, \therefore S Y^2 . P V \propto S P^3$. But it is sufficient for this conclusion, that $P V \propto S P$, and this is proved in the curve now before us, for the same reasons which were stated in the proof of Prop. IV. Cor. 8. of this Section. Taking the angle $PSQ = pSq$, we have, as stated in Case 1. $P Q : p q :: S P : S p$. Again, we have from the similar figures, $P Q : Q R :: p q : qr$. But $P V : Q P :: Q P : Q R$, and $p v : q p$

$: : qp : qr \therefore \mathrm{PV} : \mathrm{QP} : : pv : qp \therefore \mathrm{PV} : pv : : \mathrm{QP}$
$: qp : : \mathrm{SP} : \mathrm{S}p.$

Since the vel. $\propto \dfrac{1}{\mathrm{SY}}$, V . at $\mathrm{P} : v$. at $p : : \mathrm{S}y$
$: \mathrm{SY} : : \mathrm{S}p : \mathrm{SP}$. If $\mathrm{PV} = 2\,\mathrm{SP}$, the velocity in this spiral at P = velocity in circle at the same distance.

PROP. X. PROBLEM V. *Let a body revolve in an ellipse; it is required to find the law of centripetal force, tending to the centre of the ellipse.*

Let C A, C B be semi-axes of the ellipse, G P, D K other conjugate diameters ; P F, Q T perpendiculars to those diameters; Q v an ordinate to the diameter G P; and if the parallelogram QvPR is completed, the rectangle PvG will be to Qv^2,

Fig. 44.
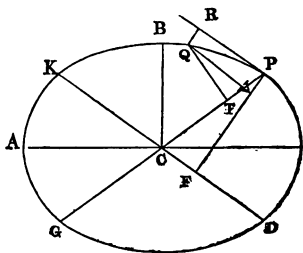
as P C^2 to C D^2; and (because of the similar triangles QvT, PCF,) Qv^2 is to QT^2, as PC^2 to PF^2; and by composition, the ratio of PvG to QT^2 is compounded of the ratio of PC^2 to CD^2, and of the ratio of PC^2 to PF^2; that is, vG is to $\dfrac{\mathrm{Q\,T^2}}{\mathrm{P}\,v}$ as PC^2 to $\dfrac{\mathrm{C\,D^2} \times \mathrm{P\,F^2}}{\mathrm{P\,C^2}}$. Substitute QR for P v, and (by Conics) B C × C A for C D × P F, also (the points P and Q continually approaching) 2 P C for v G ; and multiplying the extremes and

means together, we shall have $\dfrac{QT^2 \times PC^2}{QR}$ equal to $\dfrac{2\,BC^2 \times CA^2}{PC}$. Therefore (by Cor. 7. Prop. VI.) the centripetal force is reciprocally as $\dfrac{2\,BC^2 \times CA^2}{PC}$; that is, (because 2 $BC^2 \times CA^2$ is given) reciprocally as $\dfrac{1}{PC}$; that is, directly as the distance P C....Which was to be found.

The properties of Conic Sections here referred to, are that general characteristic of the ellipse, which makes it the genus, under which the circle and parabola are species, that the rectangle Pv . vG under the segments of a diameter : vQ^2 (semiordinate^2) : : PC^2 : CD^2, the squares of the semiconjugate diameters. Also that rectangle under semiaxes. A C . B C = parallelogram under any two semiconjugate diameters of the same ellipse, and ∴ = P F (perpendicular height of such parallelogram) × semiconjugate diameter C D. Draw out the proportions,

Pv. vG : Qv^2 : : PC^2 : CD^2 } compound these proportionals,
  Qv^2 : QT^2 : : PC^2 : PF^2 } and the results are proportional.

Pv. vG : QT^2 : : PC^4 : CD^2. PF^2, divide first and second by

Pv (= QR) and then vG : $\dfrac{QT^2}{QR}$: : PC^2 : $\dfrac{CD^2 . PF^2}{PC^2}$

∴ limit of $\dfrac{QT^2}{QR}$. PC^2 = limit of v G (i. e. 2 PC)

$\dfrac{CD^2 . PF^2}{PC^2} = \dfrac{2\,BC^2 . CA^2}{PC}$ ∴ as 2 BC^2 . CA^2 in

the same ellipse is given, F (centripetal force at P) : f . centripetal force of any other point, p in the

same ellipse : :. P C : p C. That is, the influence of the central power increases with the distance of the body on which it acts.

COR. 1. *If a body be projected in a direction making any angle with its distance from a fixed point, and be attracted to that point by a force varying as the distance, it will describe an ellipse, whose centre is the centre of force.*

Let C be the centre of force, P the point from which the body is projected in direction P Y, V the velocity, and F the force at P.

Fig. 45.
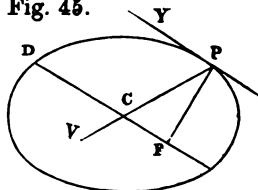
Then space (s) due to the velocity at $P = \dfrac{V^2}{2\,F}$.

In P C, produced if necessary, take $PV = 4\,s$, and draw C D parallel to P Y and $= \sqrt{\tfrac{1}{4}\,CP\,.\,PV}$. With C P, C D as semi-conjugate diameters describe an ellipse, and suppose a body revolving in it to come to P; then it is moving in the direction of the tangent at P, that is, in a line parallel to C D or in direction P Y. Also space due to velocity at $P = \frac{1}{4}$ chord of curvature at P

$$= \tfrac{1}{4} \cdot \frac{2\,C\,D^2}{C\,P} = \tfrac{1}{4}\,PV = s.$$

The force, distance, and law of force, are the same also in both cases; hence the two bodies are under the same circumstances at P, and will therefore describe the same orbit; that is, the projected body will describe an ellipse, whose centre is C.

K

If CPY be a right angle, and $s = \frac{1}{2} PC$, the orbit described will be a circle.

The object is to get the length and direction of the conjugate diameters, that will belong to the point P, of the ellipse to be described.   P Y being given as the direction of projection, will be tangent at P to the ellipse, and C being given as the centre of force, C P will be one semi-conjugate diameter, and a line drawn through C, parallel to tangent P Y, will be the other conjugate diameter, and we get its length from its being a mean proportional between 2 C P and P V the chord of curvature of the ellipse at P.   Now we know that the given velocity of projection is = the velocity that would be acquired by the action of the given central force, uniformly continued on a body from rest down $\frac{1}{4}$ of P V.   The space due to this given velocity under action of the given force is ascertained on the principle before noticed, that in such a case $\dfrac{F}{2} : S :: F^2 : V^2 \therefore S = \dfrac{V^2}{2F}$.   In P C, produced if necessary, take $PV = 4 S$.   As P V passes through the centre of the ellipse, it is = parameter, or 3d proportional of the two diameters of point P, and $\therefore 2\ CP : 2\ CD :: 2\ CD : PV$ $\therefore 4\ CD^2 = 2\ CP . PV \therefore CD = \sqrt{\frac{1}{2}\ CP . PV}$.

Cor. 2. The periodic times in all ellipses round the same centre of force in the centre, are equal. Suppose two elliptical orbits round a common centre, and that point the centre of force.   F, at

any point P, in one ellipse : f, at any point p, in other ellipse : : (by rectified Cor. 1. Prop. VI.) $\dfrac{4\,A^2\,.\,Q\,R}{C\,P^2\,.\,Q\,T^2} : \dfrac{4\,a^2\,.\,q\,r}{C\,p^2\,.\,q\,t^2}$. Substitute for $\dfrac{Q\,R}{C\,P^2\,.\,Q\,T^2}$ and $\dfrac{q\,r}{C\,p^2\,.\,q\,t^2}$ their equivalents, as proved in the demonstration of Prop. X. and we have F : f : : $\dfrac{4\,A^2\,.\,C\,P}{A\,C^2\,.\,B\,C^2} : \dfrac{4\,a^2\,.\,C\,p}{a\,C^2\,.\,b\,C^2}$. But we have seen from the Proposition, that the influence of the central force is always simply and directly in the ratio of C P or C p the distance of the body. Therefore, since in the case we are now supposing, F : f : : simply in the ratio of C P : Cp, the remaining parts of the expression for central force are constant, i. e. $\dfrac{4\,A^2}{A\,C^2\,.\,B\,C^2} = \dfrac{4\,a^2}{a\,C^2\,.\,b\,C^2}$, i. e. $\dfrac{A}{A\,C\,.\,B\,C} = \dfrac{a}{a\,C\,.\,b\,C}$. But this gives the periodical times in all such orbits equal. For the areas of ellipses are to one another as the rectangles under their axes, and the periodical times of any two orbits are by Prop. 1. Section II. as the entire areas directly, and areas dato tempore inversely. Thus per. T of Ellipse : per. t of ellipse : : $\dfrac{A\,C\,.\,B\,C}{A}$: $\dfrac{a\,C\,.\,b\,C}{a}$.

Cor. 3. To find the variation of velocity, we have Cor. 4. Prop. VI. universally F : f : : $\dfrac{V^2}{P\,V}$: $\dfrac{v^2}{p\,v}$. In the cases now before us, two ellipses round C, the same centre, F always $\propto$ P C; $\therefore$

$V^2 : v^2 :: PC . PV : pC . pv$. But $PV = \frac{2 C D^2}{P C}$,
and $pv = \frac{2 C d^2}{p c}$ $\therefore V^2 : v^2 :: C D^2 : C d^2$, and $V : v :: C D : C d$.

This also gives the periodical times equal. For take PQ and pq, simultaneous evanescent arcs; they are to one another as $V : v$, i.e. $CD : Cd$. But $PF . PQ = 2$ triangle PCQ, and for the same reasons $pf . pq = 2$ triangle pCq; but these triangles are to one another as $A : a$. $\therefore$ Per. T : per. $t :: \frac{A C . C B}{P F . P Q} : \frac{a C . C b}{p f . p q}$. But $P F = \frac{A C . C B}{C D}$, and $pf = \frac{a C . C b}{C d}$. Thus Per. $T \propto \frac{A C . C B . C D}{A C . C B . C D}$.

SCHOLIUM TO PROP. X.

If the orbit be a parabola, it may be considered an ellipse, whose farther focus is removed to an infinite distance. This removes the centre of such ellipse to an infinite distance, and this point being the centre of force of the orbit, the force acts in lines parallel to the axis; and since the difference between any two distances, $P C$ and $p C$, vanishes compared with the distances themselves, the force is invariable. The properties of the ellipse, assumed in the demonstration of this Prop. to prove that $F \propto P C$, are properties common to the hyperbola. Consequently, if a body were moving in an hyperbola, and if the constant force, drawing the body out of the tangential direction, emanated from the

centre of the hyperbola, this constant force would vary as the distance from the centre, and would be a repulsive force.

The formula $F \propto \dfrac{V^2}{PV}$ gives us in different points of the same orbit. $F \propto \dfrac{1}{PF^2 \cdot \dfrac{2\,CD^2}{PC}} \propto PC$, since

$PF =$ the SY of former cases, being $=$ perpendicular from C to tangent at P.

In Whewell's three Sections of Newton, the complete construction of the ellipse in our Cor. 1. is given.

Had the sun's place been found the common centre, and not the common focus of our planetary elliptical orbits, its attractive influence would, we see, have increased with the distance of the planet, and the periodical times of all the primary planets would have been equal. In circular orbits round a common central influence of this character, we have seen that the periodical times are equal, and the velocities $\propto$ radii. Prop. IV. Cor. 3.

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS ABOUT A CENTRE OF FORCE IN ONE OF THE FOCI.

PROP. XI. *A body revolves in an ellipse; it is required to find the force tending to the focus of the ellipse.*

LET S be the focus, P Q a small arc, Q R the subtense parallel to S P, Q T perpendicular to SP; C the centre, PC a diameter, Q v an ordinate parallel to the tangent at P, cutting S P in x.

Fig. 46.
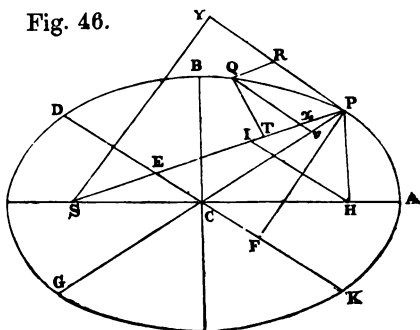
Let C D, conjugate to C P, meet S P in E ; P E is equal to the semiaxis major A C ; for if H be the other focus, and H I parallel to C E, we have S C : C H : : S E : E I ; therefore E I = E S and 2 P E = P E + E S + P E − E I = P S + P I = P S + P H, because I H being parallel to the

tangent at P, makes equal angles with PS, PH, by Conics. But $PS + PH = 2 AC$, by Conics. Hence $PE = AC$. Now we have

$QR (= \frac{1}{2} Px) : Pv :: PE : PC$ by similar triangles.

Therefore $Gv . QR : Gv . vP :: PE . PC : PC^2$,

by Conics, $Gv . vP : Qv^2 :: PC^2 : CD^2$,

ultimately $Qv^2 : Qx^2 :: 1 : 1$;

and by sim. tri. $Qx^2 : QT^2 :: PE^2 : PF^2$.

Compounding these four proportions, and observing that ultimately $Gv = 2 PC$, we have *ultimately*,

$$2 PC . QR : QT^2 :: PE^3 . PC : CD^2 . PF^2,$$
$$:: AC^3 . PC : AC^2 . BC^2,$$

hence $\dfrac{2 PC . QR}{QT^2} = \dfrac{AC . PC}{BC^2}$, $\dfrac{QR}{QT^2} = \dfrac{AC}{2 BC^2}$.

Now $\dfrac{2 BC^2}{AC} = L$ the latus rectum by Conics.

Hence $\dfrac{QR}{QT^2} = \dfrac{1}{L}$.

SP being now the distance of the body from the centre of force, the full expression for the variation of the centripetal force at different points, P and p, whether of the same orbit, or of different orbits round the same centre of force, is (by rectified Cor. 1. Prop. VI.) $\dfrac{4 A^2 . QR}{SP^2 . QT^2} = \dfrac{4 A^2}{SP^2 . L}$, hence in the same orbit $F \propto$ simply as $\dfrac{1}{SP^2}$. But in the same orbit the body may be at very different distances from the focus. Therefore we see that the law of the centripetal force is, that its action

on the body in pulling it out of its tangential direction, is more or less powerful, inversely in the duplicate ratio of the body's distance. Hence in different orbits round the same centre of force, $F : f :: S p^2 : S P^2$. Therefore as $F : f :: \dfrac{4 A^2}{S P^2 . L}$: $\dfrac{4 a^2}{S p^2 . l}$, this gives $\dfrac{4 A^2}{L} = \dfrac{4 a^2}{l}$, or $L : l :: A^2 : a^2$, as is subsequently proved in Prop. XIV.

The latus rectum, or principal parameter L, is the third proportional to the axes; and tangent to the ellipse at P, makes equal angles with the lines PS and PH from the foci to P; also PS + PH = major axis.

The formula $F \propto \dfrac{V^2}{PV}$ makes F in the same orbit $\propto \dfrac{1}{PV . SY^2}$, but $SY^2 = \dfrac{SP^2 . CB^2}{CD^2}$, and PV (through focus) $= \dfrac{2CD^2}{CA}$, hence $F \propto \dfrac{CA}{2CB^2 . SP^2}$ $\propto \dfrac{1}{L . SP^2}$.

PROP. XII. *A body moves in a hyperbola: it is required to find the force tending to the focus.*

As before (fig. 47.) let C A, C B be the semiaxes major and minor, PG, DK two conjugate diameters; S the focus, Q xv an ordinate, parallel to the tangent at P, and meeting S P in x; Q R the subtense parallel to S P.

Let S P meet C D in E; P E is equal to the semiaxes major; for if H be the other focus, and if, in the figure, H I were drawn, meeting S P

Fig. 47.
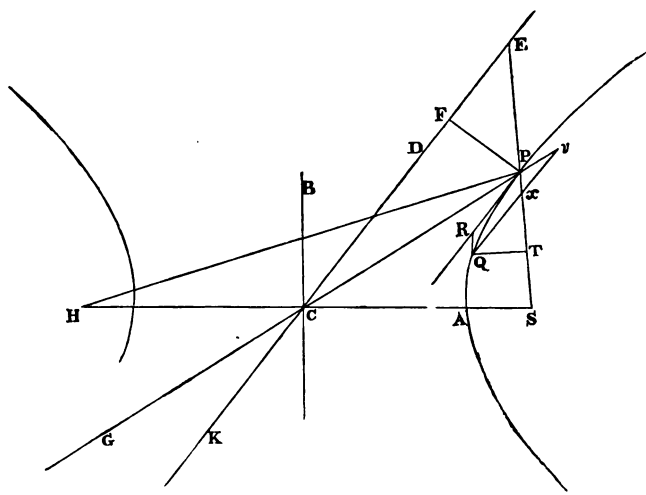
produced in the point I, we should have S C : C H :: S E : E I.   Therefore E I = E S, and 2 P E = E I + E P − (E S − E P) = I P − S P = H P − S P, [because H I being parallel to the tangent at P, makes equal angles with P I, P H, by Conics.] But H P − S P = 2 A C.   Hence P E = A C. And we have

QR ($=$ Px) : Pv :: PE : PC, by similar triangles.

Therefore Gv . QR : Gv . vP :: PE . PC : PC^2,
by Conics,   Gv . vP :   Qv^2 ::   PC^2 : CD^2,
   ultimately,   Qv^2 :   Qx^2 ::   1   : 1;
and by sim. tri.   Qx^2 : QT^2 ::   PE^2 : PF^2.

Compounding, we have *ultimately* (Gv being then = 2 P C),

2 PC . QR : QT^2 :: PE^3 . PC : CD^2 . PF^2,

   :: AC^3 . PC : AC^2 . BC^2, by Conics.

Whence $\frac{QR}{QT^2} = \frac{AC}{2BC^2} = \frac{1}{L}$.

S P being now the distance of the body from the centre of force, the full expression for the variation of the centripetal force $= \dfrac{4\,A^2 \cdot Q\,R}{S\,P^2 \cdot Q\,T^2} = \dfrac{4\,A^2}{S\,P^2 \cdot L}$, as noticed in Prop. XI. in case of the ellipse; hence the force varies inversely as the square of the distance at different points, in the same hyperbola, or in different hyperbolas round the same focus, for the reasons in case of the ellipse; and $\therefore \dfrac{4\,A^2}{L}$ is constant. In similar manner it may be shewn, if the body move in the opposite hyperbola, it must be acted on by forces tending from, instead of to, the centre of force, (the focus S,) acting repulsively, and which vary inversely as the square of the distance.

PROP. XIII. *Let a body move in a parabola: it is required to find the force tending to the focus of the Figure.*

Let S be the focus, A the vertex, P any point, Q R the subtense parallel to S P, Q T perpendicular to S P, P G parallel to the axis, Q x v parallel to the tangent; S N perpendicular to the tangent.
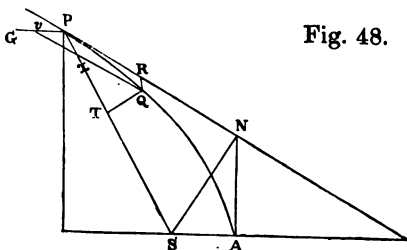
Fig. 48.

Then P G, P S make equal angles with the tangent, by Conics, to which $Q\,x\,v$ is parallel: therefore $x\,P\,v$ is an isosceles triangle ; and $P\,v = P\,x = Q\,R$. Now by Conics, $4\,S\,P\,.\,P\,v = Q\,v^2$; or $4\,S\,P\,.\,Q\,R = Q\,x^2$ ultimately, (Lemma VII. Cor. 2.)

But $Qx^2 : QT^2 :: SP^2 : SN^2 :: SP : SA$; (Parab.)
whence $4\,SP\,.\,QR : QT^2 :: SP : SA$.

$$\text{And } \frac{Q\,R}{Q\,T^2} = \frac{1}{4\,S\,A}.$$

S P being the distance of the body from the centre of force, the full expression for the variation of the centripetal force (rectified Cor. 1. Prop. VI. as in the two preceding cases) $= \dfrac{4\,A^2\,.\,Q\,R}{S\,P^2\,.\,Q\,T^2} = \dfrac{4\,A^2}{S\,P^2\,.\,L}$. Hence the force varies inversely as the square of the distance at different points, in the same parabola, or in different parabolas round the same focus, for the reasons given in the ellipse, and $\dfrac{4\,A^2}{L}$ is constant.

(Prop. VI. Cor. 3. rectified). $F \propto \dfrac{4\,A^2}{SY^2\,.\,PV}$, but $S\,Y^2 = S\,P\,.\,S\,A$, and $P\,V = 4\,S\,P$, $\therefore F \propto \dfrac{4\,A^2}{4\,S\,A\,.\,S\,P^2}$.

Cor. 1. If a body be projected at a given distance from a centre of force which $\propto$ (distance)$^{-2}$, and in a direction making a finite angle with the distance, it will describe a Conic Section.

Fig. 49.

Let S be the
centre of force, P
the point, and PY
the direction of
projection, F =
force at P, then
if s be the space

Fig 50.

due to the velocity of projection, $s = \dfrac{(\text{velocity})^2}{2\,\text{F}}$, and

is ∴ known,
on the prin-
ciple that $\dfrac{\text{F}}{2}$
$: s :: \text{F}^2 : v^2,$
$\therefore s = \dfrac{v^2}{2\,\text{F}}.$

Fig. 51.

From the data we know S to be the centre of
force, and one focus of the Conic Section; also
Z P Y to be a tangent to the section at P. To
find H, the other focus, we must find the length
and direction of H P. Now P V, chord of curva-
ture to the section at P, through S, is known =
4 s. Now whether the section be an ellipse or

hyperbola, $\text{P V} = 4\,\dfrac{(\frac{1}{2}\ \text{conjugate diameter})^2}{\text{major axis}} =$

$\dfrac{4\,\text{S P}\,.\,\text{H P}}{\text{S P} \pm \text{H P}},\ \therefore \text{H P}\,.\,4\,\text{S P} = \text{P V}\,.\,(\text{S P} \pm \text{H P}),$

$\therefore \text{H P}\,.\,(4\,\text{S P} \pm \text{P V}) = \text{P V}\,.\,\text{P S}.\ \therefore \text{H P} =$

$\dfrac{\text{P V}\,.\,\text{P S}}{4\,\text{S P} - \text{P V}} = \dfrac{\frac{\text{P V}}{4}\,.\,\text{P S}}{\text{S P} - \frac{\text{P V}}{4}} = \dfrac{s\,.\,\text{P S}}{\text{S P} - s}\ \therefore \text{H P : P S}$

$:: s : \text{P S} - s.$ If s be less than P S, H P is
positive, and ∴ is on the same side of the tangent

with P S. Consequently, draw P H on that side, making the angle Z P H = angle S P Y, and with foci S and H, and axis major = S P + P H, describe an ellipse : a body revolving in this ellipse round the given centre of force in S, will have, on coming to P, the velocity specified in the data, and will be moving in direction P Y. If s be greater than P S, H P = $\frac{s \cdot P S}{S P - s}$ is negative, and therefore lies on the other side of the tangent. Consequently, draw P H on the other side of P Y, making the angle Y P H = angle Y P S. Then with foci S and H, and axis major H P — S P describe an hyperbola. If s = S P, H P is infinite, i. e. the other focus H is at an infinite distance : make the angle Z P H = angle S P Y, and P H will be parallel to the axis. Through S draw S A T parallel to P H, and from S draw S Y perpendicular to the tangent, and from Y draw Y A perpendicular to axis S T, and with focus S and vertex A, describe a parabola. If $s = \frac{P S}{2}$ and P S be perpendicular to P Y, the section is a circle.

Thus at a given distance from a given centre of force, which $\propto (\text{dist.})^{-2}$, a body projected in a given direction will require a less velocity of projection for the ellipse than the parabola, but greater for the hyperbola than the parabola.

Cor. 2. If the velocity, with which the body goes from its place P, is such, that in any indefinitely small moment of time the line P R may

be thereby described; and the centripetal force is such, as in the same time to move that body through the space Q R; the body will move in one of the Conic Sections; whose principal *latus rectum* is the limit, to which the quantity $\frac{Q T^2}{Q R}$ approaches, while the lines P R, Q R are continually diminished.

In these Corollaries I consider the circle as an ellipse; and I except the case, where the body descends to the centre in a right line.

PROP. XIV. THEOREM VI. *If several bodies revolve about one common centre, and the centripetal force is reciprocally in the duplicate ratio of the distance of places from the centre; I say, that the principal* latera recta *of their orbits are in the duplicate ratio of the areas, which the bodies, by radii drawn to the centre, describe in the same time.*

For (by Cor. 2. Prop. XIII.) the *latus rectum* L is equal to the limit, to which the quantity $\frac{Q T^2}{Q R}$ approaches, while the distance of P and Q is continually diminished. But the small line Q R,

Fig. 52.

in a given time, is as the generating centripetal force; that is, (by supposition) reciprocally as S P². Therefore $\frac{Q T^2}{Q R}$ is as Q T² × S P²; that is,

the *latus rectum* L is in the duplicate ratio of the area $QT \times SP$. Which was to be demonstrated.

Cor. Hence the whole area of the ellipse, and the rectangle under the axes, proportional to it, is in the ratio compounded of the subduplicate ratio of the *latus rectum,* and the ratio of the periodical time. For, the whole area is as the area $QT \times SP$, which is described in a given time, multiplied into the periodical time.

We have seen, in our view of Props. XI. XII. XIII. that $\frac{A^2}{L}$ was constant, which is in substance this Prop. XIV.

Also, since period^1. time $\propto \dfrac{\text{ellipse}}{\text{area dat. temp.}}$ $\therefore$ per. t.

$\propto \dfrac{AC \cdot BC}{L^{\frac{1}{2}}}$ but $L^{\frac{1}{2}} \propto \sqrt{\dfrac{2BC^2}{AC}}$ $\therefore$ per. t. $\propto \dfrac{AC \cdot BC}{\sqrt{\dfrac{2BC^2}{AC}}}$

$\propto AC^{\frac{3}{2}}$, this is the substance of Prop. XV.

PROP. XV. THEOREM VII. *The same things being supposed, I say, that the periodical times in ellipses are in the sesquiplicate ratio of their greater axes.*

For the less axis is a mean proportional between the greater axis and the *latus rectum;* and, therefore, the rectangle under the axes is in the ratio compounded of the subduplicate ratio of the *latus rectum,* and the sesquiplicate ratio of the greater axis. But this rectangle (by Cor. Prop. XIV.) is in a ratio, compounded of the subduplicate ratio of the *latus rectum,* and the ratio of the periodical

time.   Subduct from both sides the subduplicate ratio of the *latus rectum,* and there will remain the sesquiplicate ratio of the greater axis equal to the ratio of the periodical time.   Which was to be de-monstrated.   ·

Cor. Therefore, the periodical times in ellipses are the same as in circles, whose diameters are equal to the greater axes of the ellipses.

$2\,C\,B = \sqrt{\overline{2\,A\,C}} \,.\, \sqrt{}\, \mathcal{L} \,\therefore\, 2\,A\,C \,.\, 2\,C\,B = 2\,A\,C \,.$
$\sqrt{2\,A\,C} \,.\, \sqrt{}\, \mathcal{L}$; but $2\,A\,C \,.\, 2\,C\,B \propto$ period^{l}. time
$\sqrt{L}, \,\therefore\, \overline{2\,A\,C}^{\frac{3}{2}} \,\sqrt{}\, \mathcal{L} \propto$ period. t. $\sqrt{}\, \mathcal{L} \,\therefore\, \overline{2\,A\,C}^{\frac{3}{2}}$
$\propto$ period. time.

The circle is an ellipse, whose distance between its foci has indefinitely diminished; and the para-bola is an ellipse, where this distance has in-definitely increased.   In both, the rectangle under abscissæ of diameter is in given ratio to the square of the semi-ordinate.   In the parabola the further absciss is infinitely great compared with the nearer absciss, and is therefore constant, $\therefore x \propto y^2$.

If major axis of ellipse $= 2\,R$ of circle, per. t. : per. t. $: : \overline{2\,R}^{\frac{3}{2}} : \overline{2\,R}^{\frac{3}{2}}$.   In an ellipse the line from focus to vertex of minor axis $= \dfrac{\text{major axis}}{2}$ and is called the mean distance, $\therefore$ periodical times $\propto$ in sesquiplicate ratio of mean distances.

PROP. XVI. THEOREM VIII. *The same things being supposed, and right lines being drawn to the bodies, which touch the orbits; and perpendiculars being let fall on these tangents from the common focus: I say, that the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inversely, and the subduplicate ratio of the principal* latera recta *directly.* (Fig. 52.)

From the focus S draw S Y perpendicular to the tangent P R, and the velocity of the body P will be reciprocally in the subduplicate ratio of the quantity $\frac{S Y^2}{L}$. For that velocity is as the indefinitely small arc P Q described in a given moment of time; that is, (by Lem. VII.) as the tangent P R; that is, because of the proportionals P R to Q T and S P to S Y, as $\frac{S P \times Q T}{S Y}$, or as S Y reciprocally and S P × Q T directly; but S P × Q T is as the area described in a given time; that is, (by Prop. XIV.) in the subduplicate ratio of the *latus rectum*. Which was to be demonstrated.

We here see, Cor. 1. of Prop. I. that $V \propto \frac{1}{S Y}$ extended to different orbits round the same centre of force. $V \propto \frac{A}{S Y}$, without reference to the law of the force. If $F \propto (\text{distance})^{-2}$ then $A \propto \sqrt{L}$.

Cor. 1. $V \propto \frac{\sqrt{L}}{S Y}$ ∴ in any two ellipses round a common focus of force $V : v :: \frac{\sqrt{L}}{S Y} : \frac{\sqrt{l}}{S y}$, ∴ $L : l :: V^2 . S Y^2 : v^2 . S y^2$.

Cor. 2. If A and M (see Fig. 46.) are the extremities of the major axis, they are the points of greatest and least distance from focus S, and at them the curve is perpendicular to the ray, $\therefore$ S A and S M are respectively $=$ S Y and S y. Thus at A, V $\propto \frac{\sqrt{L}}{SA}$, and at M, $v \propto \frac{\sqrt{L}}{SM}$.

Cor. 3. In circles L $=$ 2 R and S Y $=$ R, $\therefore$ vel. in Conic Section at A : vel. in Θ of rad. S A $::$ $\sqrt{L} : \sqrt{2}$ S A, for here S Y $=$ S A $=$ R.

Cor. 4. (See Fig. 46.) vel. in ell. at mean distance : vel. in Θ of rad. C A $:: \frac{\sqrt{L}}{CB} : \frac{\sqrt{2CA}}{CA}$ $\left(= \frac{\sqrt{2}}{CA^{\frac{1}{2}}}\right)$, but C B $= \sqrt{CA}.\sqrt{\frac{L}{2}} \therefore$ v. of ell. : v. of $\Theta :: \frac{\sqrt{L}}{CA^{\frac{1}{2}}.\frac{L^{\frac{1}{2}}}{\sqrt{2}}} \left(= \frac{\sqrt{.2}}{CA^{\frac{1}{2}}}\right) : \frac{\sqrt{.2}}{CA^{\frac{1}{2}}} :: 1 : 1.$

By Prop. IV. Cor. 6. velocities in circles, when F $\propto$ (distance)$^{-2} \propto \frac{1}{R^{\frac{1}{2}}}$, therefore velocities in ellipses at their mean distances are to one another inversely in subduplicate ratio of the distances.

Cor. 5. If the *latera recta* are equal in orbits, the velocities are simply in the inverse ratio of the perpendiculars from focus on the tangents.

Cor. 6. In parabola (see Fig. 51.) SY$^2 =$ SP . SA $\therefore$ S Y $\propto$ S P$^{\frac{1}{2}}$. Hence at different points P and p

in the same parabola, $V : v :: \dfrac{1}{S\,Y} : \dfrac{1}{S\,y} :: \dfrac{1}{S\,P^{\frac{1}{2}}}$: $\dfrac{1}{S\,p^{\frac{1}{2}}}$. In the ellipse and hyperbola $S\,Y^2 = B\,C^2 \cdot \dfrac{S\,P}{P\,H}$; (see Fig. 46. and 47.) hence at different points P and p in the same ellipse, $V : v$ $:: \dfrac{P\,H^{\frac{1}{2}}}{S\,P^{\frac{1}{2}}} : \dfrac{p\,H^{\frac{1}{2}}}{S\,p^{\frac{1}{2}}}$; the same in the hyperbola. Now in the ellipse $SP + PH = $ a given quantity; $\therefore$ if SP increases, PH decreases. Therefore $\dfrac{H\,P}{S\,P}$ alters its value by an increase of denominator SP, more than $\dfrac{1}{S\,P}$ would alter its value by the same increase of SP; because in the latter case the numerator remains invariable, whereas in the former case the numerator HP decreases, which has the effect of lessening the value of the fraction, still more than it will be lessened by increase of the denominator SP. Thus as SP, distance of body in parabola from focus, increases, the velocity in parabola does not decrease so much as it decreases by similar increase of SP in the ellipse. In the hyperbola $SP - HP = $ given quantity; $\therefore$ if SP increases in length, HP increases by the same length, and thus the value of $\dfrac{H\,P}{S\,P}$ alters less than the value of $\dfrac{1}{S\,P}$ by the increase of SP, because the simultaneous increase of the numerator checks the decrease of value caused by the increase of the denominator. Thus as a body moves in parabola, velocity $\propto$ (distance)$^{-\frac{1}{2}}$ in ellipse velocity is more,

in hyperbola less varied than according to this ratio.

Cor. 7. To compare V, velocity in Conic Section at distance S P, with v velocity in Θ of radius $=$ S P. In parabola, $V^2 : v^2 :: \dfrac{4\,SA}{SP\,.\,SA} : \dfrac{2\,SP}{SP^2} :: \dfrac{4}{SP}$ $: \dfrac{2}{SP} :: 2\,.\,1, \therefore V : v :: \sqrt{2} : 1$. In the ellipse and hyperbola $V^2 : v^2 :: \dfrac{\dfrac{2\,BC^2}{AC}}{BC^2\,.\,\dfrac{SP}{PH}} : \dfrac{2\,SP}{SP^2} :: HP$

$: AC \therefore V : v :: \sqrt{HP} : \sqrt{AC} :: \sqrt{2\,AC \mp SP}$ $: \sqrt{AC} :: \sqrt{2 \mp \dfrac{SP}{AC}} : 1$. Thus in the ellipse V : v in less ratio than $\sqrt{2} : 1$, but in the hyperbola in a greater ratio than $\sqrt{2} : 1$. Lastly, let Ψ be the velocity in Θ of radius $\dfrac{SP}{2}$.

In parabola $V : v :: \sqrt{2} : 1,$

$ \underline{\text{but}\quad v : \Psi :: 1 : \sqrt{2}}$

$ \therefore\quad V : \Psi :: 1 : 1.$

In the ellipse and hyperbola putting $\dfrac{SP}{AC} = \mu.$

$V : v :: \sqrt{2 \mp \mu} : 1$

$\underline{v : \Psi :: 1 : \sqrt{2}}$

$V : \Psi :: \sqrt{2 \mp \mu} : \sqrt{2}.$

Thus in parabola V $= \Psi$, in ellipse less, in hyperbola greater.

Cor. 8. If radius of $\Theta = \dfrac{L}{2}$, V in Conic Section at S P distance : v of this $\Theta :: \dfrac{L}{2} : SY.$

Cor. 9. If Ψ be the velocity in Θ of radius S P, then by Prop. IV. Cor. 6.

$$v : \Psi :: \sqrt{\overline{SP}} : \sqrt{\tfrac{L}{2}} \qquad \left\{ \begin{array}{l} \text{compound these proportion-} \\ \text{als of these 2 Cors. 8. and 9.} \end{array} \right.$$

and $\therefore$ V : Ψ :: $\sqrt{\overline{SP}} \times \sqrt{\tfrac{L}{2}}$: SY, but our third term $=$ mean proportional between S P and $\tfrac{L}{2}$.

We have thus a general formula for comparing velocity in Conic Section round focus with velocity in circle at same distance.

SCHOLIUM.

We have in these two Sections been considering the action of forces that are constantly drawing or propelling bodies towards a common centre, and thus constantly drawing them from the rectilinear directions, in which the vis inertiæ of matter would constrain them to move. And the proportion between their actual efforts for preserving a rectilinear motion, and the efforts of the central forces pulling them to a centre, may be such as to constitute a balance, and to cause them to continue revolving round the centre. We see this taking place in our planetary system.

Immense masses of matter are on this principle circulating round a central mass of matter far more immense, and the centripetal forces are that simple attraction of matter towards matter, which Newton recognised in the fall of the apple, as the cause confining the moon in her orbit, and the planets in their several orbits. In the curvilinear motions

we have been considering, we have measured the central force in any case by the velocity it generates towards the centre, and we ascertain that velocity by resolving the actual motion, or space passed over by the body, into two motions, one due to the vis inertiæ, (viz. the small portion of the tangent,) the other (viz. the subtense) due to the action of the central force. As the direction, and perhaps the velocity, of the body are constantly changing, it is only by approximations, and the doctrine of limits, that we can ascertain any correct measure of the centripetal velocity, so as to discover, by a legitimate comparison of the effects, whether difference of distance, or other causes, influence the actual power of the central force.

Our conclusions are confined to the accelerating force of gravity. No reference is made to the quantity of matter in the bodies acted on. In the forces brought into action on the earth's surface for producing mechanical effects, the object is to impart certain requisite momenta or quantities of motion to certain bodies. Therefore we must ascertain the quantity of matter in the body acted on, that we may know the due velocity to be imparted; and often the force itself which we employ, is the moving force exerted by some body, to which we are able to communicate velocity, and the amount of that moving force will depend on the quantity of matter, as well as velocity of the moving body. And this action

of the moving body, being competent to impart only a limited momentum, will impart a less and less velocity to the body acted on, according to the quantity of matter in that body. Whereas the attractive influence of a large central body, pervading every point of the circumambient space, acts on every particle of matter occupying those points of space. The influence of such attraction is weaker at greater distances, but at a given distance the influence is not weaker, because there are more and more points of the circumambient space occupied by particles of matter.

In considering the variation of the centripetal force, i. e. the ratio $F : f$ at points P and p, we have three cases; P and p may be points in the same orbit, or in different orbits, and those orbits round the same, or different centres of force. Where we take the velocity, generated in time indefinitely small by the action of centripetal force, as measure of that force, i. e.

$$2 \text{ limit } \frac{\text{space through which F draws the body}}{T},$$

our expressions and conclusions (if we retain the full original expressions, without casting off constant quantities,) apply to all three cases, as we are thus comparing causes by their real legitimate effects. Thus Prop. IV. and its two first Corollaries give us correct ratios of $F : f$, whether in orbits round the same or different centres. The subsequent Corollaries of Prop. IV. belong properly to

orbits round the same centre, as they give the ratio F : f and V : v, if a certain supposed relation is found to exist between periodical T and radius. Again, for the same reason, the original formula, given by Prop. VI. is general. In making substitutions for any term in the original formula, you must consider how far such substitutions limit the application of the formula. Props. VII. X. and XI. give formulæ for the ratio F : f even in orbits round different centres of force, if the entire original expression for the centripetal forces be retained, and not those terms in the expression be cast away, which are constant in the same orbit, or in different orbits round the same centre of force. Props. XIV. XV. XVI. are confined to the relations between periodical times, and the relations between velocities in orbits round a common focus, when F $\propto$ (distance)$^{-2}$.

It is interesting to observe how the velocity of the planet increases in its descent from aphelion to perihelion, according to the law V $\propto \frac{1}{SY}$; because this increase of velocity enables it to resist the powerfully increasing attraction of the sun, so as not to be drawn in upon it, but to wend its course round the sun, and to recede from it. And again we should observe how the velocity of the receding planet begins to lessen, and continues lessening, so that the sun's attraction, though rapidly decreasing, is competent to bring the planet round at its aphelion. Drawing also

at any different points in the elliptic orbit, the
direction in which the sun's attraction is pulling
the planet, and the direction in which the planet's
inertia is at the same moment urging it, you will
see how these two directions do more converge or
more diverge, so that by a cooperating, or an anta-
gonistic action of the two influences, the planet's
motion is duly accelerated, or duly retarded.

ON ANGULAR VELOCITY.

The angular velocity of a body, moving in an
orbit round a centre of force S, (fig. 52. p. 142.)
is measured by the angle uniformly described by
S P round S in 1′, in the same manner as linear
velocity is measured by the line uniformly de-
scribed in 1′. If the angular motion of S P be
not uniform, the angular velocity at any point is
measured by the angle, which would be described
in 1′, if the angular motion of S P were to con-
tinue uniform for that time. Hence if the angular
motion be not uniform, and P S Q be the angle
described in T′ after leaving P, the angular velocity

$$= \text{limit} \ \frac{\text{angle P S Q}}{\text{T}},$$

for this is the angle which would be described in
1′, if the angular motion at P were to continue
uniform for that time.

PROP. *If a body be moving in any orbit round a centre of force* S, *the angular velocity at any point* P

$$= \frac{2\,A}{S\,P^2}.$$

Let P S Q be the angle described in T″, with centre S and radius S Q, describe a circular arc cutting SP in T, and draw S Y perpendicular to the tangent at P; then the triangle P T Q may be considered as ultimately rectilinear, and similar to S Y P, hence, since an angle varies as an arc subtending it directly, and radius inversely,

$$\angle^r \text{ vel. at } P = \text{limit } \frac{\angle \text{ PSQ}}{T} = \frac{QT}{SQ \cdot T}$$

$$= \text{limit } \frac{PQ \cdot SY}{SP^2 \cdot T}, \text{ since limit } \frac{QT}{PQ} = \frac{SY}{SP}$$

(and S Q ultimately $=$ S P)

$$= \frac{SY \cdot \text{vel. at } P}{SP^2}, \text{ since limit } \frac{PQ}{T} = \text{vel. at P.}$$

$$= \frac{2\,A}{S\,P^2},$$

for the velocity at P would be measured by the line, which that velocity continued uniform would describe in a given time, and SY × that line would be equal to twice the area described by P S in that time.

Idem aliter. If the angle P S Q be indefinitely small, the angular velocity with which P S describes it round S, may be considered as continued uniform through the very small time of its description; on the same principle on which the linear velocity of a body has been considered

uniform through an evanescent arc. Therefore, if psq be another evanescent angle described in the given very small time, by ps round another centre of force s, the angular velocities at P and p will be to one another as the angles (fig. 36. p. 106.)

$$\text{PSQ} : psq :: \frac{\text{QT}}{\text{PS}} : \frac{qt}{ps} :: \frac{\text{SP} \cdot \text{QT}}{\text{SP}^2} : \frac{sp \cdot qt}{sp^3} :: \frac{2\,\text{A}}{\text{SP}^2} : \frac{2\,a}{sp^2}$$

$\therefore$ in the same orbit, where A and a are equal, the angular velocity of the planet's motion round the sun $\propto$ inversely as the square of the planet's distance from the sun. The angular motion of a planet round the upper focus of its elliptical orbit is found to be nearly uniform, and this serves in many cases as a useful approximation towards finding the true anomaly.

If the student has attentively read these three Sections with my illustrations, he will have gained, I trust, a satisfactory conception of the general principles of Physical Astronomy; and in Brinkley's Elements of Astronomy, composed for the University of Dublin, he will find a useful course of Plane and Practical Astronomy.

In the Corollaries at the close of Prop. XIII. Newton had said, I consider the Circle as an Ellipse, and I except the case, where the body descends to the centre in a right line. This case is afterwards considered in Props. XXXIII. XXXVI. and XXXVIII.

It is brought under the general head of the ellipse by indefinitely diminishing the minor axis

of the ellipse, and thus making the area at length narrow, or shrink into coincidence with the major axis.

Force varying as (distance)$^{-2}$. To find the time of motion and the velocity acquired by a body falling through a given space from rest. (Props. XXXIII. and XXXVI.)

Let S be the centre of force, A the point from which the body begins to fall.

Let A B P be a semi-ellipse, focus S and axis major A S B; A D B a semi-circle, whose diameter is ASB; and suppose a body revolving in the ellipse round the focus S to come to P; bisect A B in O, draw D P C perpendicular to A B, and join O P, O D.
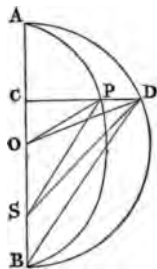
Fig. 53.

Then the time through AP $\propto$ area ASP $\propto$ area A S D, and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point P with C, or the body is moving in the straight line A C; the point B also coincides with S, since AS . SB = ($\frac{1}{2}$ axis minor)^2; and since space due to velocity at A = $\frac{1}{4}$ chord of curvature at A through S = $\frac{1}{4}$ latus rectum $= \frac{(\text{axis minor})^2}{4 \text{ A B}} = 0$, the body begins to move from rest at A.

Hence time from rest through AC ∝ area ABD, ∴ time through A C : time through A B (= $\frac{1}{2}$ periodic time in ellipse) : : area A B D : semi-circle A B D. Now to get the expression for variation of periodic time in ellipse, suppose R O minor axis, and set down the proportion P′ (= periodic time in seconds) : 1′ :: area of ellipse (= π. A O . R O)

: area in 1′ = A; but A ∝ L$^{\frac{3}{2}}$, and L$^{\frac{3}{2}}$ ∝ $\dfrac{R O}{\sqrt{. A O}}$

∴ P′ : 1′ :: π. A O . R O : $\dfrac{R O}{\sqrt{. A O}}$, ∴ P′ ∝ π . A O$^{\frac{3}{2}}$,

∴ time through AC ∝ $\dfrac{\pi . A O^{\frac{3}{2}}}{2 . \frac{1}{2} \pi A O^{2}}$, $\frac{1}{2}$ AO . (AD + CD)

∝ $\dfrac{A O^{\frac{3}{2}} . (A D + C D)}{2}$, for area ABD = $\frac{1}{2}$ AO . AD

+ $\dfrac{C D . B O}{2}$, and $\dfrac{A O^{\frac{3}{2}}}{A O^{2}} = \dfrac{1}{A O^{\frac{1}{2}}}$.

S being given the centre of force, and the body being considered to fall from rest from A, we begin with supposing S the lower focus of an ellipse in which the body is moving, and therefore that B the lower extremity of the major axis lies below S, while H the upper focus lies between vertex A, and O centre of ellipse. The axis minor diminishing without limit, H becomes coincident with A, and B with S, and the space due to velocity at A becomes = 0, and O moving upwards AS = 2 AO. In different points of the same descent from A to S time through A C ∝ A D + C D, i. e. as ordinate of a cycloid, whose axis is A B, vertex A, and generating ⊖ A D B, for such ordinate to C in the axis = AD + DC.

If a body be moving in ellipse A P B round focus S, we know from Cor. 6. Prop. XVI. velocity $\propto \dfrac{\sqrt{H P}}{\sqrt{S P}}$. But where ellipse coincides with major axis, P coincides with C, and H with A ; $\therefore$ H P = A C, for the same reason S P = S C or B C. $\therefore$ In fall from A to S velocity at C $\propto \dfrac{\sqrt{A C}}{\sqrt{S C}} \propto \dfrac{A C}{C D}$. If arc A D = θ, $\dfrac{A C}{S C} = \dfrac{1 - \cos. \theta}{1 + \cos. \theta}$ = tangent^2 of $\dfrac{\theta}{2}$ $\therefore$ vel. at C $\propto$ tangent of arc $\dfrac{A D}{2}$.

If we are considering the more simple case of the perpendicular fall from a given point A to S the centre of force, and are enquiring the ratio of the times of descent, and the velocities acquired at different points in this fall, A C is known, and $\therefore$ the arc A D is known, being that arc of the semi-circle on A S, whose versed sine is A C, and the time, due to fall down A C, varies as the length of arc A D + the length of C D the sine of that arc, and the velocity acquired $\propto \dfrac{A C \text{ the versed sine}}{C D \text{ the right sine}} \Big\}$ of arc A D. If we are supposing two falls from two different points or heights, A and B to the same centre of force S, we then have two distinct semi-ellipses, each with its auxiliary semi-circle. And the time of fall to any point P in one fall, would be to the time of fall to any point p in the other fall, as $\Big(\dfrac{A \, S \frac{1}{2}}{2} \cdot \dfrac{(A D + C D)}{2} \Big) : \Big(\dfrac{B \, S \frac{1}{2}}{2} \cdot \Big(\dfrac{B d + c d}{2} \Big) \Big)$.

Again, the ratio of the velocities acquired at P and p would want an addition to our formula,

since the proportion $V \propto \dfrac{\sqrt{HP}}{\sqrt{SP}}$ holds good at different points in the same ellipse, but not in different ellipses.

Let us now suppose that

Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest. (Prop. XXXVIII. of Newton.)

Let S be the centre of force, A the place from which the body begins to fall: on A B = 2 A S describe a semi-ellipse A P B, and a semi-circle A D B, and let a body moving in the ellipse come to P. Draw D P C perpendicular to A B, and join S P, S D.

Fig. 54.
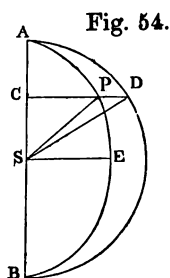
Then time through A P $\propto$ area A S P $\propto$ area A S D, and this being true, whatever be the axis minor of the ellipse, will be true when it is diminished without limit, in which case the body will be at C, having fallen from rest at A,

$\therefore$ time through A C $\propto$ area A S D

$\therefore \dfrac{\text{time through A C}}{\text{time through A S} \left(= \tfrac{1}{4} \text{ periodic time in a circle} \right)}$

$= \dfrac{\text{sector A S D}}{\tfrac{1}{4} \text{ area of circle}};$

$\therefore$ time through A C $\propto \dfrac{\text{time through A S . sector A S D}}{\tfrac{1}{4} \text{ area of circle with rad'. A S}},$

but sector A S D $= \tfrac{1}{2}$ rad. A S, arc A D, and

periodic[l]. times of all ellipses, round the centre
where force $\propto$ distance are equal; $\therefore$ in falls
from different heights to S, time through A S is
constant; and in all such cases time through A C
$\propto \dfrac{\frac{1}{2} A S . A D}{\frac{1}{4} \pi . A S^2}$, and at different points in the same
fall, $t \propto A S . A D \propto A D$ the arc of the quadrant
on A S as rad[s]. whose versed sine is the given
space A C. The times of the entire falls from B
or A, or other points, to S are all equal, as the
period[l]. times of all ellipses round S as the common
centre are equal. Consequently, if a large circular
hole were bored from the surface to the centre of
the earth, all bodies, if let fall from the surface of
the earth, or from any point in this hole below the
surface, would reach the centre in equal time.
For in such cases the attraction of the earth's
matter above the falling body, combined with the
attraction below, give a force which $\propto$ as the
distance from the earth's centre. Again, let S E
be the semi-axis minor; then vel. at P $\propto$ semi-
conjugate at P, in all these ellipses round the
same centre. But the semi-conjugate at P =
$\sqrt{A S^2 + S E^2 - S P^2}$: vel. at C $\propto \sqrt{A S^2 - S C^2}$
$\propto$ C D, the right sine of the arc A D, which repre-
sents the time, A C being versed sine.

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In putting together my Compilation, I have
borrowed largely from Mr. Evans's very useful
Edition of these three Sections. And, since my

completion of this little work, I have been made acquainted with the existence of Mr. Goodwin's Elementary Course of Mathematics. This Book promises, from its title and plan, to be a very valuable Guide and Text-Book for our Mathematical Studies in Oxford. The Preface and the table of Contents lead one to expect a Course of Algebra, Trigonometry, and Conic Sections, confined within the bounds which I have ventured in my Preface to advocate, viz. not opening a field of wide unlimited research in these departments, but concentrating attention on those points, portions, and properties, an acquaintance with which is necessary to a sound scientific progress in the four great branches of Natural Philosophy. The student is there led on to a comprehensive knowledge of those branches. But the three first Sections of the Principia are interposed as the stepping-stone to Astronomy, to give a clear understanding of the physical principles which regulate the laws and motions in our Planetary System. I rejoice to see, from Mr. Goodwin's Preface, that the wisdom of the Cambridge Senate enforces a Geometrical Course of Mathematical reading, by the order, that the three first days of the eight days of their Public Examination be occupied by Examinations in the Elementary parts of Statics, Dynamics, Hydrostatics, Optics, and Astronomy, to be treated Geometrically, without the Differential Calculus.

When we go on and read the additional order of

the Senate, that in all these subjects, examples
and questions shall be introduced into these
Examinations, we must be aware, that to pass
through such an ordeal with credit, will require
a respectable proficiency in Mathematical and
Physical knowledge, a proficiency equal to the
maximum that will probably be attainable, with
their limited opportunities, by the majority of our
Oxford Students. The maxim, therefore, which
their experience in the effects and tendencies of
Mathematical Studies has dictated to the Cam-
bridge Senate, should not be lost upon us,—that
the mind should be trained to certain habits and
proficiency in Geometrical reasoning, and by that
instrument should have become conversant with
the elementary parts of Natural Philosophy, before
it endeavours to carry on its researches by the
Differential and Integral Calculus.

For the Grace of the Senate, which regulates
the Examination of Candidates for Mathematical
Honours, thus orders:

" 1. That after the first three days there shall
be an interval of eight days, and that on the
seventh of those days the Moderators and Ex-
aminers shall declare what persons have so ac-
quitted themselves as to deserve Mathematical
Honours.

" 2. That those who are declared to have so
acquitted themselves, and no others, be admitted
to the Examination in the higher parts of Mathe-
matics."

I conclude with the earnest hope, that those habits of enlightened and enlarged reflection, which our predominance of Classical Studies should generate, will prevent our Physical Students from resting with complacent conceit in their disclosures and detection of the laws that regulate the phæ-nomena and productions of nature, as if their conclusions were the ultimatum and perfection of knowledge. Let their minds open to the thought, that the conclusions of Science are only a partial discovery of the laws by which the Almighty has ordered certain agencies to be operating, but that we are in the dark as to the individual constitution or nature of such agents or agencies; into this mystery our human faculties cannot penetrate. Perhaps they may, when enlarged in a future and higher existence. In the several departments of natural knowledge, we may be simplifying our systems by detecting higher and more compre-hensive principles; but there will always remain a higher principle or agency still beyond, that is to us a mystery. This being the case in those de-partments of knowledge which seem on a level with our capacity, is it philosophical or consistent with analogy, that we should find offence at mysteries in that knowledge which is evidently above our powers of understanding or conception, the nature of an Eternal Almighty Creator, and His dealings with His creatures ?